Solutions 3

2026

1. Obtain a three-term asymptotic approximation for the integral

$$I(\varepsilon) = \int_{\varepsilon}^{1} \sin(\varepsilon t)\mathrm{d}t,$$

as $\varepsilon \to 0$.

Solution:

$$\begin{aligned} I(\varepsilon) = \int_{\varepsilon}^{1} \sin(\varepsilon t)\mathrm{d}t &= \int_{\varepsilon}^{1} \left\{ \varepsilon t - \frac{(\varepsilon t)^3}{3!} + \frac{(\varepsilon t)^5}{5!} + O((\varepsilon t)^7) \right\} \mathrm{d}t, \quad \varepsilon \to 0 \\ &= \left[\frac{\varepsilon t^2}{2} - \frac{\varepsilon^3 t^4}{24} + \frac{\varepsilon^5 t^6}{720} + O(\varepsilon^7) \right]_{\varepsilon}^{1}, \quad \varepsilon \to 0 \\ &= \frac{\varepsilon}{2} - \frac{13\varepsilon^3}{24} + \frac{\varepsilon^5}{720} + O(\varepsilon^7), \quad \varepsilon \to 0. \end{aligned}$$

2. Obtain a three-term asymptotic approximation for the integral

$$J(\alpha) = \int_{0}^{\infty} \frac{5x - 3\alpha}{x + \alpha} e^{-x} \mathrm{d}x$$

where $\alpha \gg 1$ is a large parameter.

Solution: Let $\varepsilon = \frac{1}{\alpha}$; since α is large, ε is small. Manipulating the integrand we note that

$$J(\varepsilon) = \int_{0}^{\infty} \frac{5x - \frac{3}{\varepsilon}}{x + \frac{1}{\varepsilon}} e^{-x} \mathrm{d}x = \int_{0}^{\infty} \frac{5\varepsilon x - 3}{\varepsilon x + 1} e^{-x} \mathrm{d}x.$$

Now, recall that $\frac{1}{1+\varepsilon x} = 1 - \varepsilon x + \varepsilon^2 x^2 - \ldots$. It follows that

$$J(\varepsilon) = \int_{0}^{\infty} (-3 + 8\varepsilon x - 8\varepsilon^2 x^2 + \ldots) e^{-x} \mathrm{d}x \simeq -3 \int_{0}^{\infty} e^{-x} \mathrm{d}x + 8\varepsilon \int_{0}^{\infty} x e^{-x} \mathrm{d}x - 8\varepsilon^2 \int_{0}^{\infty} x^2 e^{-x} \mathrm{d}x.$$

Computing the integrals (the latter two by parts) gives that

$$J(\varepsilon) = -3 + 8\varepsilon - 16\varepsilon^2 + O(\varepsilon^3), \quad \varepsilon \to 0,$$

or in the original variables:

$$J(\alpha) = -3 + \frac{8}{\alpha} - \frac{16}{\alpha^2} + O\left(\frac{1}{\alpha^3}\right), \quad \alpha \to \infty.$$

3. Use Laplace's method to obtain a leading order approximation of the integral

$$\int_0^{2\pi} (t^2+1)e^{-\lambda(2+\sin t)}\mathrm{d}t$$

as $\lambda \to \infty$.

Let $g(t) = 2 + \sin t$ and note that over $(0, 2\pi)$, g is minimised at $3\pi/2$, where $g(3\pi/2) = 1$. We rewrite the integral as

$$\mathcal{K}(\lambda) = \int_0^{2\pi} (t^2+1)e^{-\lambda(2+\sin t)}\mathrm{d}t = \int_0^{2\pi} (t^2+1)e^{-\lambda[(2+\sin t)+1-1]}\mathrm{d}t = e^{\lambda}\int_0^{2\pi} (1+t^2)e^{-\lambda[\sin t+1]}\mathrm{d}t.$$

Note that $sint + 1 \geq 0$ for all $t \in (0, 2\pi)$, with equality iff $t = 3\pi/2$. For large λ, the main contribution to the integral therefore comes from a small neighbourhood of $t = 3\pi/2$. That is,

$$\mathcal{K}(\lambda) \sim \int_{\frac{3\pi}{2}-\varepsilon}^{\frac{3\pi}{2}+\varepsilon} (1+t^2)e^{-\lambda[\sin t+1]}\mathrm{d}t, \qquad \lambda \to \infty,$$

for some small ε. Since this integration is performed over a very small neighbourhood of $t = 3\pi/2$, the factor of $(1 + t^2)$ in the integrand can be approximated to leading order as $(1 + (3\pi/2)^2)$, giving

$$\mathcal{K}(\lambda) \sim \left(1+\frac{9\pi^2}{4}\right)e^{-\lambda}\int_{\frac{3\pi}{2}-\varepsilon}^{\frac{3\pi}{2}+\varepsilon} e^{-\lambda[\sin t+1]}\mathrm{d}t, \qquad \lambda \to \infty.$$

We now Taylor-expand the exponent near $t = 3\pi/2$, thus obtaining

$$\mathcal{K}(\lambda) \sim \left(1+\frac{9\pi^2}{4}\right)e^{-\lambda}\int_{\frac{3\pi}{2}-\varepsilon}^{\frac{3\pi}{2}+\varepsilon} \exp\left(-\lambda\left[-1+\frac{1}{2}\left(t-\frac{3\pi}{2}\right)^2+\ldots+1\right]\right)\mathrm{d}t, \qquad \lambda \to \infty.$$

Simplifying the exponent gives

$$\mathcal{K}(\lambda) \sim \left(1+\frac{9\pi^2}{4}\right)e^{-\lambda}\int_{\frac{3\pi}{2}-\varepsilon}^{\frac{3\pi}{2}+\varepsilon} \exp\left(-\frac{\lambda(t-\frac{3\pi}{2})^2}{2}\right)\mathrm{d}t, \qquad \lambda \to \infty.$$

Now, the integrand is very small for large λ, except in a small neighbourhood of $3\pi/2$. We can therefore enlarge the interval of integration to $(-\infty, \infty)$, and so

$$\mathcal{K}(\lambda) \sim \left(1+\frac{9\pi^2}{4}\right)e^{-\lambda}\int_{-\infty}^{\infty} \exp\left(-\frac{\lambda(t-\frac{3\pi}{2})^2}{2}\right)\mathrm{d}t, \qquad \lambda \to \infty.$$

Finally, make the substitution $x = \sqrt{\frac{\lambda}{2}}(t - \frac{3\pi}{2})$ (so $\mathrm{d}x = \sqrt{\frac{\lambda}{2}}\mathrm{d}t \Leftrightarrow \mathrm{d}t = \sqrt{\frac{2}{\lambda}}\mathrm{d}x$), and recall that $\int\limits_{-\infty}^{\infty} \exp(-x^2)\mathrm{d}x = \sqrt{\pi}$, giving

$$\mathcal{K}(\lambda) \sim \left(1 + \frac{9\pi^2}{4}\right) e^{-\lambda}\sqrt{\frac{2\pi}{\lambda}}, \qquad \lambda \to \infty.$$

4. Find a two-term asymptotic approximation of a periodic solution of the Duffing equation

$$\ddot{x} + 4x - 2\varepsilon x^3 = 0,$$

where the system is released from rest with an initial amplitude A, and ε is a small parameter.

We use the Linstedt-Poincaré method. Let us first consider the limit case ($\varepsilon = 0$). In this case the Duffing equation reduces to $\ddot{x} + 4x = 0$, which has solutions of the form

$$x_0(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

Note that this limit case solution has frequency $\omega_0 = 2$. According to the Linstedt-Poincaré method, we seek periodic solutions with (unknown) frequency ω. We make the change of variables $\tau = \omega t$; in the new scaled time variable the Duffing equation becomes

$$\omega^2 \frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} + 4x - 2\varepsilon x^3 = 0. \tag{1}$$

We seek an asymptotic approximation for x in the form

$$x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots,$$

and similarly expand the unknown frequency ω as $\omega = \omega_0 + \varepsilon\omega_1 + \dots$ By substituting these expressions into (1) and collecting terms in the same degree of ε, we obtain

$$\omega_0^2 \frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} + \omega_0^2 x_0 = 0, \tag{2}$$

$$\omega_0^2 \frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + 2\omega_0\omega_1 \frac{\mathrm{d}^2 x_0}{\mathrm{d}\tau^2} + 4x_1 - 2x_0^3 = 0. \tag{3}$$

Solving (2) with initial conditions $x_0(0) = A$ (initial amplitude A) and $x'(0) = 0$ (released from rest i.e. zero initial velocity), we obtain

$$x_0(\tau) = A\cos\tau.$$

Substituting this into (3) yields

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + x_1 = \omega_1 A \cos\tau + \frac{A^3}{2}\left(\frac{1}{4}\cos(3\tau) + \frac{3}{4}\cos\tau\right),$$

where we have used the identity $\cos^3\tau \equiv \frac{1}{4}(3\cos\tau + \cos(3\tau))$. The above equation has solution (by complementary/particular solution method)

$$x_1(\tau) = \left(\frac{3A^2}{16} + \frac{A\omega_1}{2}\right)\tau\sin\tau - \frac{A^3}{64}\cos(3\tau) + C_3\cos\tau + C_4\sin\tau.$$

We note that the term involving $\tau \sin \tau$ is a *secular term*, that is, a term that grows unboundedly as $\tau \to \infty$. Its presence prevents our solution from being periodic. This gives a condition on ω_1 for the solution to be periodic, since the coefficient $3A^2/16 + A\omega_1/2$ must be zero. Thus for periodic solutions, $\omega_1 = -\frac{3A^2}{8}$.

To preserve initial conditions we impose the following conditions on x_1:

$$x_1(0) = 0, \quad x_1'(0) = 0.$$

These enable us to find the constants C_3 and C_4, which are given by

$$C_3 = \frac{A^3}{64}, \quad C_4 = 0.$$

Thus $x_1(\tau) = -\frac{A^3}{64}\cos(3\tau) + \frac{A^3}{64}\cos\tau$. Combining the above, our two-term asymptotic approximation is given by

$$x(\tau) = A\cos\tau + \varepsilon\frac{A^3}{64}(\cos\tau - \cos(3\tau)) + O(\varepsilon^2), \quad \varepsilon \to 0,$$

where $\tau = \omega t$ and

$$\omega = 2 - \varepsilon\frac{3A^2}{8} + O(\varepsilon^2), \quad \varepsilon \to 0.$$