S: seen similar
B: bookwork
U: unseen

MA34210 : Asymptotic Methods in Mechanics 2023

SECTION A

Q1 Landau Symbols S/B

a) If there exist $c, \delta > 0$ such that $|f(x)| \leq c|g(x)|$ for all $x > \delta$, then $f(x) = O(g(x))$, $x \to \infty$. 2

b) For $x > 0$, $\left|x^2 \cos\frac{2}{x}\right| = x^2\left|\cos\frac{2}{x}\right| \leq x^2$. It follows that $x^2\cos\frac{2}{x} = O(x^2)$, $x \to \infty$. 3

c) Let $c > 0$ (small) be given. If there exists $\delta = \delta(c)$ such that $|f(x)| \leq c|g(x)|$ for all $x > \delta$, then $f(x) = \bar{o}(g(x))$, $x \to \infty$. 2

d) Let $c > 0$ (small) be given. Then for $x > \dfrac{1+\sqrt{1+8c}}{2c}$, $|x+2| \leq c|x^2|$. It follows that $x + 2 = \bar{o}(x^2)$, $x \to \infty$. 4

e) If $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$, then $f(x) \sim g(x)$, $x \to \infty$. 2

f) (e.g.) $\dfrac{xe^x}{x+1}$. 2

g) $n = 6$. 2

17 Q1

Q2 Regularly perturbed algebraic problem S

a) The limit problem corresponding to $\varepsilon = 0$ is $x^3 - 3x^2 - 4x = 0$

$\iff x(x+1)(x-4) = 0$.

This has roots $x = -1$, $x = 0$, $x = 4$. 4

b) Regular perturbation. The limit problem has the same number of roots as the original problem by the fundamental theorem of algebra. 3

c) Let $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$, $\varepsilon \to 0$. Then $x^2 = x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)$, $\varepsilon \to 0$, and $x^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2)$, $\varepsilon \to 0$.

Substitute into the cubic:

$$x_0^3 + 3\varepsilon x_0^2 x_1 - 3x_0^2 - 6\varepsilon x_0 x_1 - 4x_0 - 4\varepsilon x_1 + \varepsilon + O(\varepsilon^2) = 0, \quad \varepsilon \to 0$$ 3

Compare coefficients of ε^0: $x_0^3 - 3x_0^2 - 4x_0 = 0 \Rightarrow x_0 \in \{-1, 0, 4\}$ 3

Compare coefficients of ε^1: $3x_0^2 x_1 - 6x_0 x_1 - 4x_1 + 1 = 0$

$$\Rightarrow x_1 = \frac{1}{4 + 6x_0 - 3x_0^2} \Rightarrow x_1 \in \left\{-\tfrac{1}{5}, \tfrac{1}{4}, -\tfrac{1}{20}\right\} \text{ respectively.}$$ 3

Combining the above,
$$x = \begin{cases} -1 - \frac{\varepsilon}{5} + O(\varepsilon^2), & \varepsilon \to 0, \\ 0 + \frac{\varepsilon}{4} + O(\varepsilon^2), & \varepsilon \to 0, \\ 4 - \frac{\varepsilon}{20} + O(\varepsilon^2), & \varepsilon \to 0. \end{cases}$$ 1

17 Q2

Q3 a) A singularly perturbed problem's solution cannot be uniformly approximated by an asymptotic expansion $\sum_{n=0}^{N} \delta_n(\varepsilon)\, a_n(x)$, $\varepsilon \to 0$ (where $(\delta_n)_{n=0}^{N}$ is an asymptotic sequence). For algebraic problems, this exhibits itself as roots being lost when using the regular perturbation ansatz. 2

Singularly perturbed algebraic problem

S/B

b) Singularly perturbed roots tend towards infinity in some direction in $\mathbb{C}$ as $\varepsilon \to 0$, whereas the regular perturbation ansatz tends to x_0, a constant. 2

c) (i) 1 root (singularly perturbed). 2

(ii) 5 roots (all regularly perturbed). 2

(iii) 3 roots (one regularly perturbed, two singularly perturbed). 2

10 Q3

Q4 a)

Integral approximation via Taylor series

S

$$f(x) = \frac{1}{2}\int_0^x \frac{e^z \cos z}{1-\frac{z}{2}}\, dz$$

$$= \frac{1}{2}\int_0^x \left(\left(1+z+\frac{z^2}{2!}\right)\left(1-\frac{z^2}{2!}\right)\left(1+\frac{z}{2}+\frac{z^2}{4}\right)+O(z^3)\right) dz \qquad 3$$

$$= \frac{1}{2}\int_0^x \left(\left(1+z\right)\left(1+\frac{z}{2}+\frac{z^2}{4}\right)+O(z^3)\right) dz$$

$$= \frac{1}{2}\int_0^x \left(1+\frac{3}{2}z+\frac{3}{4}z^2+O(z^3)\right) dz$$

$$= \frac{1}{2}\left[z+\frac{3z^2}{4}+\frac{z^3}{4}+O(z^4)\right]_{z=0}^{z=x} \qquad 3$$

$$= \frac{1}{2}\left(x+\frac{3x^2}{4}+\frac{x^3}{4}+O(x^4)\right)$$

$$= \frac{x}{2}+\frac{3x^2}{8}+\frac{x^3}{8}+O(x^4), \quad x \to 0. \qquad 4$$

b) (i) True.

(ii) True.

(iii) False. 3

13 Q4

Q5. a) First order, linear 2

Regularly perturbed ODE

S

b) Let $u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + O(\varepsilon^3)$, $\varepsilon \to 0$.

Substitute into the ODE:

$$u_0'(x) + \varepsilon u_1'(x) + \varepsilon^2 u_2'(x) - 2\varepsilon u_0(x) - 2\varepsilon^2 u_1(x) + O(\varepsilon^3) = 1 + \varepsilon \cos x, \quad \varepsilon \to 0.$$

4

Compare coefficients of ε^0:

$u_0'(x) = 1 \Rightarrow u_0(x) = x + A$, and $u_0(0) = 1 \Rightarrow A = 1$, so $\boxed{u_0(x) = x + 1.}$

2

Compare coefficients of ε^1:

$$u_1'(x) - 2u_0(x) = \cos x \Rightarrow u_1'(x) = 2x + 2 + \cos x$$

$$\Rightarrow u_1(x) = x^2 + 2x + \sin x + B,$$

and $u_1(0) = 0 \Rightarrow B = 0$, so $\boxed{u_1(x) = x^2 + 2x + \sin x.}$

2

Compare coefficients of ε^2:

$$u_2'(x) - 2u_1(x) = 0 \Rightarrow u_2'(x) = 2x^2 + 4x + 2\sin x$$

$$\Rightarrow u_2(x) = \frac{2x^3}{3} + 2x^2 - 2\cos x + C,$$

and $u_2(0) = 0 \Rightarrow C = 2$, so $\boxed{u_2(x) = \frac{2x^3}{3} + 2x^2 - 2\cos x + 2}$.

2

Thus $\boxed{u(x) = x + 1 + \varepsilon(x^2 + 2x + \sin x) + \varepsilon^2\left(\frac{2x^3}{3} + 2x^2 - 2\cos x + 2\right) + O(\varepsilon^3), \ \varepsilon \to 0.}$

1

13 Q5

SECTION A
70

SECTION B

Q6. Let $g(z) = z - \ln z$. Then $g'(z) = 1 - \frac{1}{z} = 0$ if $z = 1$.

Laplace's method for integral transform approximation

S

Moreover, $g''(z) = z^{-2}$, so $g''(1) = 1 > 0$. The function g therefore takes its minimum value over $(0, \infty)$ at $z = 1$, where $g(1) = 1$.

This inspires us to rewrite:

$$\Gamma(n+1) = n^{n+1} \int_0^\infty e^{-n(z - \ln z - 1)} e^{-n} \, dz$$

$$= n^{n+1} e^{-n} \int_0^\infty e^{-n(z - \ln z - 1)} \, dz.$$

3

Now, since $z - \ln z - 1 \geqslant 0$ for all $z \in (0, \infty)$, with equality only if $z = 1$. Note that as $n \to \infty$, the integrand is very small indeed, except in a small neighbourhood of $z = 1$.

Consequently, $\Gamma(n+1) \sim n^{n+1} e^{-n} \int_{1-\varepsilon}^{1+\varepsilon} e^{-n(z - \ln z - 1)} \, dz$ as $n \to \infty$, for some small ε.

3

We now Taylor-expand $g(z)$ near $z = 1$: $g(z) \sim g(1) + \frac{g'(1)}{1!}(z-1) + \frac{g''(1)}{2!}(z-1)^2$

$$= 1 + \tfrac{1}{2}(z-1)^2$$

so $\Gamma(n+1) \sim n^{n+1} e^{-n} \int_{1-\varepsilon}^{1+\varepsilon} e^{-n(\frac{1}{2}(z-1)^2)} \, dz.$

Note that as $n \to \infty$, the integrand is very small except in a small neighbourhood of $z = 1$, so we can extend the integration interval:

$$\Gamma(n+1) \sim n^{n+1} e^{-n} \int_{-\infty}^{\infty} e^{-\frac{n}{2}(z-1)^2} \, dz.$$

3

Let $w = \sqrt{\frac{n}{2}}(z-1) \Rightarrow dw = \sqrt{\frac{n}{2}} \, dz$, so $\Gamma(n+1) \sim n^{n+1} e^{-n} \int_{-\infty}^{\infty} e^{-w^2} \, dw \sqrt{\frac{2}{n}}$

$$= n^{n+1} e^{-n} \sqrt{\frac{2\pi}{n}}$$

$$= \sqrt{2\pi} \, n^{n+\frac{1}{2}} e^{-n}, \quad n \to \infty,$$

as required.

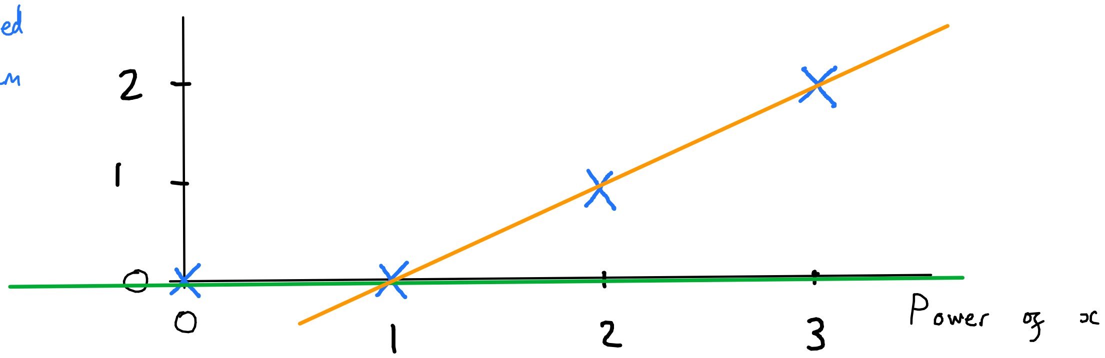
12 Q6

3

Q7. a) Power of ε

Singularly perturbed algebraic problem

S



2

Green line has slope 0, corresponds to regularly perturbed root – seek using regular perturbation ansatz $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$, $\varepsilon \to 0$. (*) 1

Orange line has slope 1; use singular perturbation ansatz $x = \varepsilon^{-1} z(\varepsilon)$, $z(0) \neq 0$. (†) 1

b) Regularly perturbed root: substituting (*) into the cubic gives

$$-5\varepsilon x_0^2 + 6x_0 + 6\varepsilon x_1 + 1 + O(\varepsilon^2) = 0, \quad \varepsilon \to 0.$$

Compare coefficients of ε^0: $6x_0 + 1 = 0 \Rightarrow x_0 = -\frac{1}{6}$.

Compare coefficients of ε^1: $-5\left(-\frac{1}{6}\right)^2 + 6x_1 = 0 \Rightarrow x_1 = \frac{5}{216}$.

Combining, $\boxed{x^{(1)} = -\frac{1}{6} + \frac{5\varepsilon}{216} + O(\varepsilon^2), \quad \varepsilon \to 0.}$ 2

Singularly perturbed roots: substituting (†) into the cubic gives

$$\varepsilon^{-1} z^3 - 5\varepsilon^{-1} z^2 + 6\varepsilon^{-1} z + 1 = 0$$

$$\Leftrightarrow \quad z^3 - 5z^2 + 6z + \varepsilon = 0$$ 1

Let $z = z_0 + \varepsilon z_1 + O(\varepsilon^2)$, $\varepsilon \to 0$. Then

$$z_0^3 + 3\varepsilon z_0^2 z_1 - 5z_0^2 - 10\varepsilon z_0 z_1 + 6z_0 + 6\varepsilon z_1 + \varepsilon + O(\varepsilon^2) = 0, \quad \varepsilon \to 0.$$

Compare coefficients of ε^0: $z_0^3 - 5z_0^2 + 6z_0 = 0$

$\Leftrightarrow z_0(z_0^2 - 5z_0 + 6) = 0$

$\Leftrightarrow z_0(z_0 - 2)(z_0 - 3) = 0$

$\Rightarrow z_0 \in \{2, 3\}$. ($z_0 \neq 0$ by (†)) 2

Compare coefficients of ε^1: $3z_0^2 z_1 - 10 z_0 z_1 + 6z_1 + 1 = 0$

$$\iff z_1 = \frac{-1}{3z_0^2 - 10z_0 + 6}$$

For $z_0 = 2$: $\quad z_1 = \frac{-1}{12 - 20 + 6} = \frac{1}{2}$.

For $z_0 = 3$: $\quad z_1 = \frac{-1}{27 - 30 + 6} = -\frac{1}{3}$.

Combining,

$$\boxed{\begin{aligned} x^{(1)} &= \frac{2}{\varepsilon} + \frac{1}{2} + O(\varepsilon), \quad \varepsilon \to 0, \\ x^{(2)} &= \frac{3}{\varepsilon} - \frac{1}{3} + O(\varepsilon), \quad \varepsilon \to 0. \end{aligned}}$$

2

11 Q7

Q8. a) Let $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$, $\varepsilon \to 0$.

Lindstedt – Poincaré Method in unfamiliar setting

U

Substitute into the ODE, giving:

$$x_0''(t) + \varepsilon x_1''(t) + \varepsilon x_0^2(t) x_0'(t) - \varepsilon x_0'(t) + 4x_0(t) + 4\varepsilon x_1(t) + O(\varepsilon^2) = 0, \quad \varepsilon \to 0$$

2

Compare coefficients of ε^0:

$$\begin{cases} x_0''(t) + 4x_0(t) = 0 \\ x_0(0) = a_0 \\ x_0'(0) = 0 \end{cases} \implies \boxed{x_0(t) = a_0 \cos(2t).}$$

2

Compare coefficients of ε^1:

$$x_1''(t) + x_0^2(t) x_0'(t) - x_0'(t) + 4x_1(t) = 0$$

$$\Rightarrow x_1''(t) + 4x_1(t) = x_0'(t) - x_0^2(t) x_0'(t)$$

$$= -2a_0 \sin(2t) + 2a_0^3 \cos^2(2t) \sin(2t)$$

This is of the form given in the hint, with $A=0$, $B=-2a_0$, $C = 2a_0^3$. Hence,

$$x_1(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{2a_0^3 - 8a_0}{16} t\cos(2t) + \frac{2a_0^3}{128} \sin(6t).$$

$$= c_1 \cos(2t) + c_2 \sin(2t) - \tfrac{1}{8}(a_0^3 - 4a_0) t \cos(2t) - \tfrac{1}{64} a_0^3 \sin(6t).$$

3

The condition $x_1(0) = 0$ implies $0 = c_1$.

Since $x_1'(0) = 2c_2 - \frac{a_0^3 - 4a_0}{8} - \frac{6a_0^3}{64} = 0$, we have $c_2 = \frac{7a_0^3 - 16a_0}{64}$.

Thus $\boxed{x_1(t) = \frac{7a_0^3 - 16a_0}{64} \sin(2t) + \tfrac{1}{8}(4a_0 - a_0^3) t \cos(2t) - \tfrac{1}{64} a_0^3 \sin(6t).}$

2

b) Secular terms become unbounded as their argument (often time) tends to infinity. This is often physically unrealistic, and makes expansions valid only on a small time period.

$\frac{1}{8}(4a_0 + a_0^3)t\cos(2t)$ is secular. 3

c) Let $\tau = \omega t$ so $\frac{dx}{d\tau} = \frac{1}{\omega}\frac{dx}{dt}$ and $\frac{d^2x}{d\tau^2} = \frac{1}{\omega^2}\frac{d^2x}{dt^2}$.

The ODE becomes $\omega^2 \frac{d^2x}{d\tau^2} + \varepsilon(x^2-1)\omega\frac{dx}{d\tau} + 4x = 0$. (*) 3

Hereafter, primes will denote derivatives w.r.t. τ.

Let $x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + O(\varepsilon^2)$, $\varepsilon \to 0$, and $\omega = \omega_0 + \varepsilon\omega_1 + O(\varepsilon^2)$, $\varepsilon \to 0$.

Note that $\omega_0 = 2$, since the limit problem (previously considered in (a)) has period π.

Equation (*) becomes: $(4 + 4\varepsilon\omega_1)(x_0'' + \varepsilon x_1'') + 2\varepsilon(x_0^2 - 1)x_0' + 4x_0 + 4\varepsilon x_1 + O(\varepsilon^2) = 0$ 3

Equate coefficients of ε^0: $4x_0'' + 4x_0 = 0 \Rightarrow x_0(\tau) = A\cos\tau + B\sin\tau$.

Initial conditions: $x_0(0) = a_0 \Rightarrow A = a_0$

while $\left.\frac{dx}{dt}\right|_{t=0} = 0 \Rightarrow B = 0$. Thus $\boxed{x_0(\tau) = a_0\cos\tau.}$ 3

Equate coefficients of ε^1: $4x_1'' + 4x_1 + 4\omega_1 x_0'' + 2x_0^2x_0' - 2x_0' = 0$

$$\Rightarrow x_1'' + x_1 = \frac{1}{2}x_0' - \frac{1}{2}x_0^2x_0' - \omega_1 x_0''$$
$$= -\frac{a_0}{2}\sin\tau + \frac{1}{2}a_0^3\cos^2\tau\sin\tau + \omega_1 a_0\cos\tau.$$

which is of the form seen in the hint with $A = \omega_1 a_0$, $B = -\frac{a_0}{2}$, $C = \frac{1}{2}a_0^3$.

Thus $\boxed{x_1(\tau) = c_1\cos\tau + c_2\sin\tau - \frac{1}{8}\left(\frac{1}{2}a_0^3 - 2a_0\right)\tau\cos\tau + \frac{\omega_1 a_0}{2}\tau\sin\tau - \frac{a_0^3}{64}\sin(3\tau).}$ 3

Initial conditions: $x_1(0) = 0 = c_1$,

and $x_1'(0) = 0 = c_2 - \frac{1}{8}\left(\frac{1}{2}a_0^3 - 2a_0\right) - \frac{3a_0^3}{64} \Rightarrow c_2 = \frac{a_0}{64}(7a_0^2 - 16)$.

Hence $x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + O(\varepsilon^2)$, $\varepsilon \to 0$

$$= a_0\cos\tau + \varepsilon\left(\frac{a_0}{64}(7a_0^2 - 16)\sin\tau - \frac{1}{8}\left(\frac{1}{2}a_0^3 - 2a_0\right)\tau\cos\tau + \frac{\omega_1 a_0}{2}\tau\sin\tau - \frac{a_0^3}{64}\sin(3\tau)\right)$$
$$+ O(\varepsilon^2), \quad \varepsilon \to 0.$$

For periodic solutions, we require $\frac{\omega_1 a_0}{2} = 0$ and $\frac{a_0^3}{16} - \frac{a_0}{4} = 0$.

The latter gives $\frac{a_0}{16}(a_0^2 - 4) = 0 \Rightarrow \boxed{a_0 = 2}$ since $a_0 > 0$, whence $\boxed{\omega_1 = 0.}$ 3

27 Q8

SECTION B

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