MA34210 2022/2 Solutions

Q1 a) For $x \in (0,1)$, $|x^5 \cos x| = x^5 |\cos x| \leq x^5 \leq x^4$. It follows that $x^5 = O(x^4)$, $x \to 0$. 3

b) Let $c > 0$ (small) be given. Then for $x > \frac{3+\sqrt{9+4c}}{2c}$, $|1+3x| \leq cx^2$. It follows that $1+3x = o(x^2)$, $x \to 0$. 4

c) (for example) $\phi(x) = 6x^{-5}$. 1

8

Q2 Let $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$, $\varepsilon \to 0$. Then $x^2 = x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)$, $x^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2)$, $\varepsilon \to 0$.

Substitute into cubic: $x_0^3 + 3\varepsilon x_0^2 x_1 - 3x_0^2 - 6\varepsilon x_0 x_1 - 13x_0 - 13\varepsilon x_1 + 15 + \varepsilon + O(\varepsilon^2) = 0$. 4

Compare coefficients of $\varepsilon^0$: $x_0^3 - 3x_0^2 - 13x_0 + 15 = 0 \iff (x_0+3)(x_0-1)(x_0-5) = 0$,

with roots $x_0^{(1)} = -3$, $x_0^{(2)} = 1$, $x_0^{(3)} = 5$. 3

Compare coefficients of $\varepsilon^1$: $3x_0^2 x_1 - 6x_0 x_1 - 13x_1 = -1 \Rightarrow x_1 = \dfrac{1}{13 + 6x_0 - 3x_0^2}$,

so $x_1^{(1)} = -\frac{1}{32}$, $x_1^{(2)} = \frac{1}{16}$, $x_1^{(3)} = -\frac{1}{32}$. 3

Combining the above,
$$x = \begin{cases} -3 - \varepsilon/32 + O(\varepsilon^2), & \varepsilon \to 0, \\ 1 + \varepsilon/16 + O(\varepsilon^2), & \varepsilon \to 0, \\ 5 - \varepsilon/32 + O(\varepsilon^2), & \varepsilon \to 0. \end{cases}$$
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12

Q3 a) Second order, linear. 2

b) Let $u(x) = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^2)$, $\varepsilon \to 0$. Then $u''(x) = u_0''(x) + \varepsilon u_1''(x) + O(\varepsilon^2)$, $\varepsilon \to 0$.

Substitute into the ODE: $u_0''(x) + \varepsilon u_1''(x) - 12\varepsilon u_0(x) = x^2 + 2\varepsilon x + O(\varepsilon^2)$, $\varepsilon \to 0$. 3

Compare coefficients of $\varepsilon^0$: $u_0''(x) = x^2 \Rightarrow u_0'(x) = \frac{x^3}{3} + A \Rightarrow u_0(x) = \frac{x^4}{12} + Ax + B$

and $u_0(0) = 0 \Rightarrow B = 0$, $u_0(1) = 1 \Rightarrow A = \frac{11}{12}$, so $u_0(x) = \frac{1}{12}(x^4 + 11x)$.

3

Compare coefficients of $\varepsilon^1$: $u_1''(x) - 12u_0(x) = 2x$

$\Rightarrow u_1''(x) = x^4 + 13x$

$\Rightarrow u_1'(x) = \frac{x^5}{5} + \frac{13x^2}{2} + A$

$\Rightarrow u_1(x) = \frac{x^6}{30} + \frac{13x^3}{6} + Ax + B$,

and $u_1(0) = 0 \Rightarrow B = 0$, $u_1(1) = 0 \Rightarrow \frac{1}{30} + \frac{13}{6} + A = 0 \Rightarrow A = -\frac{11}{5}$. 4

Thus $u_1(x) = \frac{x^6}{30} + \frac{13x^3}{6} - \frac{11x}{5}$, and so $u(x) = \frac{1}{12}(x^4 + 11x) + \varepsilon\left(\frac{x^6}{30} + \frac{13x^3}{6} - \frac{11x}{5}\right) + O(\varepsilon^2)$, $\varepsilon \to 0$.

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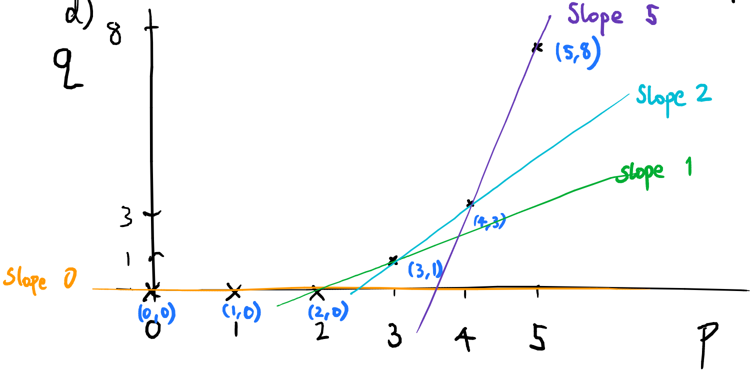
14

Q4. a) 5 roots 1

b) $8+7x-x^2=0$ has two roots in $\mathbb{C}$. 2

c) Singularly perturbed; the limit problem has fewer roots than the original. 3

d)




(Plot $(p,q)$ $\forall$ $p,q$ s.t. $C_{p,q}\neq 0$ when equation written as

$$\sum_{p,q} C_{p,q}\,\varepsilon^q x^p = 0 \Big)$$ 5

e) The line of slope zero corresponds to the two regularly perturbed roots. Other roots should be sought in the forms:

$x=\varepsilon^{-1}b(\varepsilon)$, $\quad b(0)\neq 0$, $\quad$ (one root);

$x=\varepsilon^{-2}b(\varepsilon)$, $\quad b(0)\neq 0$, $\quad$ (one root);

$x=\varepsilon^{-5}b(\varepsilon)$, $\quad b(0)\neq 0$, $\quad$ (one root). 5

f) The root approximated by $x=\varepsilon^{-5}b(\varepsilon)$ will diverge most rapidly. 1

g) • For regularly perturbed roots, solve $8+7x-x^2=0 \iff (8-x)(x+1)=0$
$\Rightarrow x=-1,+8$ to leading order)

• For $x\sim\varepsilon^{-1}b_0$, terms in $x^2$, $x^3$ are balanced, so solve $-\varepsilon^{-2}b_0^2+2\varepsilon^{-2}b_0^3=0$
$\Rightarrow b_0=\frac{1}{2}$ (since $b_0\neq 0$). Therefore leading order approx. is $x\sim\frac{1}{2\varepsilon}$, $\varepsilon\to 0$.

• For $x\sim\varepsilon^{-2}b_0$, terms in $x^3$, $x^4$ are balanced, so solve $2\varepsilon^{-5}b_0^3-5\varepsilon^{-5}b_0^4=0$
$\Rightarrow b_0=\frac{2}{5}$ (since $b_0\neq 0$). Therefore $x\sim\frac{2}{5\varepsilon^2}$, $\varepsilon\to 0$.

• For $x\sim\varepsilon^{-5}b_0$, terms in $x^4$, $x^5$ are balanced, so solve $-5\varepsilon^{-17}b_0^4+2\varepsilon^{-17}b_0^5=0$
$\Rightarrow b_0=\frac{5}{2}$ (since $b_0\neq 0$). Therefore $x\sim\frac{5}{2\varepsilon^5}$, $\varepsilon\to 0$. 8

25

PTO

Q5. Taylor series approximations for small $t$:

$$\sin(2t) = 2t + O(t^3), \quad t \to 0$$

$$e^t = 1 + t + \frac{t^2}{2!} + O(t^3), \quad t \to 0$$

$$\frac{1}{1-t} = 1 + t + t^2 + O(t^3), \quad t \to 0.$$

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Thus
$$t(\sin(2t) + e^t)\frac{1}{1-t} = t\left(2t + 1 + t + \frac{t^2}{2}\right)(1 + t + t^2) + O(t^4), \quad t \to 0$$
$$= \left(t + 3t^2 + \frac{t^3}{2}\right)(1 + t + t^2) + O(t^4), \quad t \to 0$$
$$= t + t^2 + t^3 + 3t^2 + 3t^3 + \frac{t^3}{2} + O(t^4), \quad t \to 0$$
$$= t + 4t^2 + \frac{9}{2}t^3 + O(t^4),$$

5

and so
$$\int_0^x t(\sin(2t) + e^t)\frac{1}{1-t}\,dt = \int_0^x \left(t + 4t^2 + \frac{9}{2}t^3 + O(t^4)\right)dt$$
$$= \left[\frac{t^2}{2} + \frac{4t^3}{3} + \frac{9t^4}{8} + O(t^5)\right]_0^x$$
$$= \frac{x^2}{2} + \frac{4x^3}{3} + \frac{9x^4}{8} + O(x^5), \quad x \to 0.$$

(so $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = \frac{4}{3}$, $c_4 = \frac{9}{8}$).

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11

# SECTION B

Q6 a) Secular terms are terms that grow unboundedly as their argument, $t$ usually, tends to infinity. This is often physically unrealistic and makes approximations valid only for small time 3

b) Consider the limit case $(\varepsilon = 0)$ : $\ddot{x}_0 + 3x_0 = 0 \Rightarrow x_0(t) = C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)$.
This has frequency $\omega_0 = \sqrt{3}$.
We seek periodic solutions with unknown frequency $\omega = \sqrt{3} + \varepsilon\omega_1 + O(\varepsilon^2)$, $\varepsilon \to 0$.
Let $\tau = \omega t$. The Duffing equation becomes:

$$\omega^2 \frac{d^2x}{d\tau^2} + 3x + \varepsilon x^3 = 0 \qquad (*)$$

5

Seek $x(\tau)$ in the form $x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + O(\varepsilon^2)$, $\varepsilon \to 0$.
Substituting into $(*)$ gives (with primes denoting $\tau$-derivatives):

$$(3 + 2\sqrt{3}\varepsilon\omega_1 + O(\varepsilon^2))(x_0'' + \varepsilon x_1'' + O(\varepsilon^2)) + 3x_0 + 3\varepsilon x_1 + \varepsilon x_0^3 + O(\varepsilon^2) = 0, \quad \varepsilon \to 0$$

Terms in $\varepsilon^0$: $3(x_0'' + x_0) = 0$, $x_0(0) = A$, $x_0'(0) = 0 \Rightarrow x_0(\tau) = A\cos\tau$. 2

Terms in $\varepsilon^1$: $3(x_1'' + x_1) = 2\sqrt{3}\omega_1 A\cos\tau - A^3\cos^3\tau$, $x_1(0) = 0$, $x_1'(0) = 0$.

$$\Rightarrow x_1'' + x_1 = \frac{2\omega_1 A}{\sqrt{3}}\cos\tau - \frac{A^3}{12}\cos(3\tau) - \frac{A^3}{4}\cos\tau \quad \text{(by given identity)}$$

$$= \underbrace{\left(\frac{2\omega_1 A}{\sqrt{3}} - \frac{A^3}{4}\right)}_{\alpha}\cos\tau + \underbrace{\left(-\frac{A^3}{12}\right)}_{\beta}\cos(3\tau)$$

$$\Rightarrow x_1(\tau) = \frac{\alpha}{2}\tau\sin\tau - \frac{\beta}{8}\cos(3\tau) + C_1\cos\tau + C_2\sin\tau \quad \text{(by given hint)}.$$

Initial conditions give $C_1 = \frac{\beta}{8} = -\frac{A^3}{96}$ and $C_2 = 0$.
To eliminate secular terms for a periodic solution, $\alpha \equiv 0 \Rightarrow \omega_1 = \frac{A^2\sqrt{3}}{8}$.

Combining: $x(\tau) = A\cos\tau + \frac{\varepsilon A^3}{96}(\cos(3\tau) - \cos\tau) + O(\varepsilon^2)$, $\varepsilon \to 0$, and

$$\omega = \sqrt{3} + \frac{\varepsilon A^2\sqrt{3}}{8} + O(\varepsilon^2), \quad \varepsilon \to 0. \qquad (\dagger)$$

6

c) A spring with restoring force $F(x)$ is compliant if its stiffness $F'(x)$ decreases with increasing displacement.
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d) We see from $(\dagger)$ that increasing $A$ results in a decrease in frequency.
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20

Q7 a) $g(0) = 2$ 2

b) $\mathcal{P}(\lambda) = \int_{-1}^{1} \cos(t) e^{-\lambda(2+\sin^2 t)} dt = e^{-2\lambda} \int_{-1}^{1} \cos(t) e^{-\lambda \sin^2 t} dt.$

Note $\sin^2 t \geq 0$ with equality iff $t=0$. It follows that the main contribution to the integral $\mathcal{P}(\lambda)$ comes from a small neighbourhood of $0$ as $\lambda \to \infty$. Thus for some small $\varepsilon$,

$$\mathcal{P}(\lambda) \sim e^{-2\lambda} \int_{-\varepsilon}^{\varepsilon} \cos(t) e^{-\lambda \sin^2 t} dt, \quad \lambda \to \infty \qquad 4$$

On $(-\varepsilon, \varepsilon)$, $\cos(t) \sim 1$, so $\mathcal{P}(\lambda) \sim e^{-2\lambda} \int_{-\varepsilon}^{\varepsilon} e^{-\lambda \sin^2 t} dt, \lambda \to \infty$.

Since $\sin^2 t \sim t^2 - \frac{2t^4}{3} + O(t^6)$, $t \to 0$, $\mathcal{P}(\lambda) \sim e^{-2\lambda} \int_{-\varepsilon}^{\varepsilon} e^{-\lambda t^2} dt$

The integrand is vanishingly small outside $(-\varepsilon, \varepsilon)$, so $\mathcal{P}(\lambda) \sim e^{-2\lambda} \int_{-\infty}^{\infty} e^{-\lambda t^2} dt.$

Let $x = \sqrt{\lambda}\, t$. Then $\int_{-\infty}^{\infty} e^{-\lambda t^2} dt = \int_{-\infty}^{\infty} e^{-x^2} \frac{1}{\sqrt{\lambda}} dx = \lambda^{-\frac{1}{2}} \pi^{\frac{1}{2}}$.

It follows that $\mathcal{P}(\lambda) \sim e^{-2\lambda} \lambda^{-\frac{1}{2}} \pi^{\frac{1}{2}}, \quad \lambda \to \infty$ 6

c) No (or at least not without adaptation), since $\sin^2 t$ is $\pi$-periodic and so over $(-1, 4)$ would attain two local minima as $\sin^2(0) = \sin^2(\pi) = 0$. Consequently the main contribution to $\mathcal{P}$ would come from two, not one, points. 3

15

Q8 a) $\frac{dx}{dt} + x = 0 \iff e^t \frac{dx}{dt} + e^t x = 0 \iff \frac{d}{dt}\{e^t x\} = 0 \Rightarrow x = Ce^{-t}.$ 4

b) The problem described has initial conditions $x(0) = 0$, $\dot{x}(0) = v_0$. Applying these to the general solution from a):

$$x(0) = 0 \Rightarrow C = 0 \quad \text{(giving a trivial solution)}$$

$$\dot{x}(0) = v_0 \Rightarrow C = -v_0.$$

Both conditions cannot be satisfied simultaneously. 2

c) Let $T = \frac{t}{\varepsilon}$. Then $\frac{d}{dt} = \frac{d}{d(\varepsilon T)} = \frac{1}{\varepsilon}\frac{d}{dT}$, and so denoting $X(T,\varepsilon) = x(t,\varepsilon)$: 3

$$\frac{1}{\varepsilon}\frac{\partial^2 X}{\partial T^2} + \frac{1}{\varepsilon}\frac{\partial X}{\partial T} + X = 0 \Rightarrow \frac{\partial^2 X}{dT^2} + \frac{\partial X}{\partial T} + \varepsilon X = 0, \text{ with } X(0) = 0,\ X_T(0) = \varepsilon v_0$$

Let $X(T,\varepsilon) = \varepsilon X_1(T) + \varepsilon^2 X_2(T) + O(\varepsilon^3)$, $\varepsilon \to 0$.

Then $\varepsilon X_1'' + \varepsilon^2 X_2'' + \varepsilon X_1' + \varepsilon^2 X_2' + \varepsilon^2 X_1'' + O(\varepsilon^3) = 0, \quad \varepsilon \to 0.$

Compare coefficients of $\varepsilon^1$: $X_1''(T) + X_1'(T) = 0 \Rightarrow X_1(T) = A_1 + B_1 e^{-T}$, with $A_1 = v_0$ by I.C.s and $B_1 = -v_0$, so $X_1(T) = v_0(1 - e^{-T})$ 3

Compare coefficients of $\varepsilon^2$:

$$X_2''(T) + X_2'(T) = v_0(e^{-T} - 1),$$ which has complimentary function

$$X_{2c} = A_2 + B_2 e^{-T}.$$

Seek a particular integral in the form $aT + bTe^{-T}$.
The usual approach, with homogeneous I.C.s yields

$$X_2(T) = v_0\left(2(1-e^{-T}) - T(1+e^{-T})\right).$$

Combining, $$X(T) = \varepsilon v_0\left(1-e^{-T}\right) + \varepsilon^2 v_0\left(2(1-e^{-T}) - T(1+e^{-T})\right) + O(\varepsilon^2),$$
$\varepsilon \to 0$.

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15