

Solutions 2

1. Consider the cubic equation

$$\varepsilon^2 x^3 + \varepsilon x^2 + 4x - 3 = 0.$$

- (a) How many roots in \mathbb{C} does this equation have?
- (b) Find two-term *regularly perturbed* solutions using the straightforward asymptotic expansion $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$, $\varepsilon \rightarrow 0$.
- (c) Draw the Kruskal-Newton graph for this problem and use it to construct a singular perturbation asymptotic Ansatz.
- (d) Use the singular perturbation Ansatz from (c) to find two-term *singularly perturbed* solutions.
- (e) Evaluate all three roots numerically for the case $\varepsilon = 0.01$.

Solution:

- (a) 3 roots.
- (b) We make the usual regular perturbation Ansatz

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2), \quad \varepsilon \rightarrow 0.$$

Noting that $x^2 = x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)$ and $x^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2)$ as $\varepsilon \rightarrow 0$, the original equation becomes

$$\varepsilon x_0^2 + 4x_0 + 4\varepsilon x_1 - 3 + O(\varepsilon^2) = 0, \quad x \rightarrow 0.$$

Comparing coefficients of ε^0 and ε^1 we get

$$4x_0 - 3 = 0, \quad \text{and} \quad x_0^2 + 4x_1 = 0,$$

whence $x_0 = 3/4$ and $x_1 = -9/64$. Therefore the two-term asymptotic expansion of the regularly perturbed root is

$$x = \frac{3}{4} - \frac{9\varepsilon}{64} + O(\varepsilon^2), \quad \varepsilon \rightarrow 0.$$

- (c) The appropriate singular perturbation Ansatz is

$$x = \varepsilon^\alpha z(\varepsilon), \quad z(0) \neq 0;$$

the Kruskal-Newton graph inspires us to choose $\alpha = -1$ (the line with slope 0 corresponds to the regular perturbation problem considered in (b)). Thus our Ansatz is

$$x = \varepsilon^{-1} z(\varepsilon), \quad z(0) \neq 0.$$

(d) Substitute the singular perturbation Ansatz into the cubic equation to obtain

$$\varepsilon^{-1}z^3(\varepsilon) + \varepsilon^{-1}z^2(\varepsilon) + 4\varepsilon^{-1}z(\varepsilon) - 3 = 0.$$

Multiply throughout by ε :

$$z^3(\varepsilon) + z^2(\varepsilon) + 4z(\varepsilon) - 3\varepsilon = 0. \quad (1)$$

We now use regular perturbation methods on z . We make the regular perturbation Ansatz

$$z(\varepsilon) = z_0(\varepsilon) + \varepsilon z_1(\varepsilon) + O(\varepsilon^2), \quad \varepsilon \rightarrow 0.$$

Substituting this Ansatz into (1) and comparing coefficients of terms in different degrees of ε gives

$$z_0^3 + z_0^2 + 4z_0 = 0, \quad (2)$$

and

$$3z_0^2 z_1 + 2z_0 z_1 + 4z_1 - 3 = 0. \quad (3)$$

Since $z_0 \neq 0$, we have that $z_0^2 + z_0 + 4 = 0$ and so

$$z_0^\pm = \frac{-1 \pm i\sqrt{15}}{2}.$$

Putting these values into (3) gives that the corresponding values for z_1^\pm are

$$z_1^+ = -\frac{1}{8}(3 - \sqrt{3/5}i)$$

and

$$z_1^- = -\frac{1}{8}(3 + \sqrt{3/5}i).$$

Thus the singularly perturbed roots x^\pm are given by

$$x^+ = \frac{1}{\varepsilon} \left(\frac{-1 + i\sqrt{15}}{2} - \frac{\varepsilon}{8}(3 - i\sqrt{3/5}) + O(\varepsilon^2) \right), \quad \varepsilon \rightarrow 0,$$

and

$$x^- = \frac{1}{\varepsilon} \left(\frac{-1 - i\sqrt{15}}{2} - \frac{\varepsilon}{8}(3 + i\sqrt{3/5}) + O(\varepsilon^2) \right), \quad \varepsilon \rightarrow 0.$$

(e) The two term estimates for the three roots when $\varepsilon = 0.01$ are

$$x = 0.74859375, \quad x^+ = -50.375 - 193.7459919i, \quad x^- = -50.375 - 193.7459919i.$$

For interest (not asked), the actual solutions in this $\varepsilon = 0.01$ case are

$$x = 0.74858855, \quad x^+ = -50.374294 + 193.74687i, \quad x^- = -50.374294 - 193.74687i.$$

Pretty good accuracy for a two term approximation!

2. Consider the singularly perturbed quartic equation

$$1 - x + 7\epsilon x^2 - 3\epsilon^3 x^3 + \epsilon^7 x^4 = 0.$$

- (a) Find three-term *regularly perturbed* solutions using the straightforward asymptotic expansion.
- (b) Draw the Kruskal-Newton graph.
- (c) Which powers of ϵ would you use in the singular perturbation Ansatz for this problem?

Solution:

- (a) Substituting the usual regular perturbation Ansatz into the quartic and collecting terms in the same power of ϵ gives

$$1 - x_0 = 0,$$

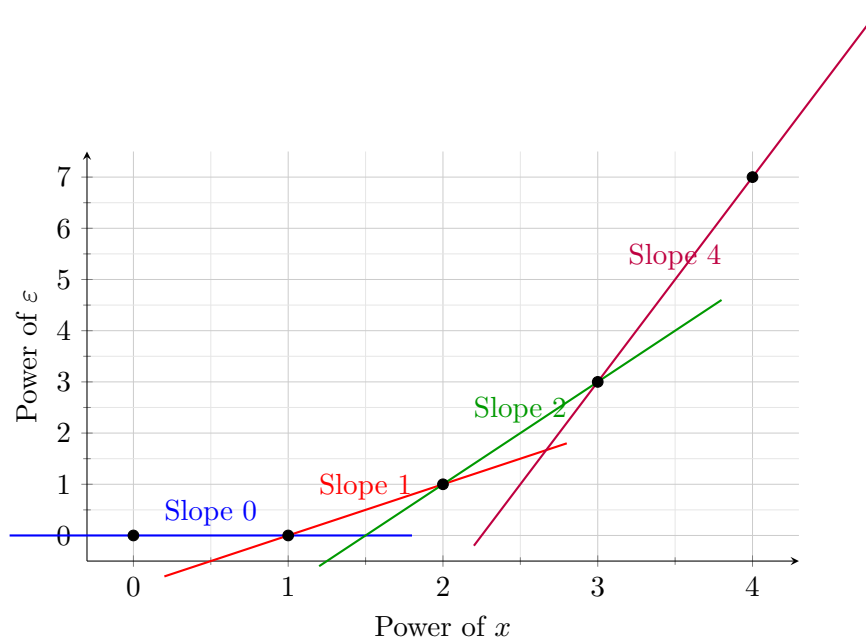
$$-x_1 + 7x_0^2 = 0.$$

and

$$-x_2 + 14x_0x_1 = 0.$$

Thus $x_0 = 1$, $x_1 = 7$ and $x_2 = 98$, so the regularly perturbed root is

$$x^{(1)} = 1 + 7\epsilon + 98\epsilon^2 + O(\epsilon^3), \quad \epsilon \rightarrow 0.$$



- (b)
- (c) Use $\alpha = 0$ (regular perturbation case considered in (a)), $\alpha = -1, -2, -4$.

3. Consider the ordinary differential equation

$$-4\mu^2 y''(x) + (1 - x^2)y(x) = 0,$$

with boundary conditions $y(0) = 2$ and $y(1) = -\frac{1}{2}$.

Find a two-term asymptotic approximation for the solution in the form

$$y(x) = y_0(x) + \mu^{-2}y_1(x) + O(\mu^{-4}), \quad \mu \rightarrow \infty,$$

assuming that $\mu \gg 1$ is a large positive parameter.

Hint: make the substitution $\varepsilon = \mu^{-2}$.

Solution:

Use the suggested substitution $\varepsilon = \mu^{-2}$ and multiply throughout by ε to get

$$-4y''(x) + \varepsilon(1 - x^2)y(x) = 0.$$

Now use straightforward asymptotics, i.e. write $y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$ and substitute into the equation. This yields

$$-4y_0'' - 4\varepsilon y_1'' - 4\varepsilon^2 y_2'' + \varepsilon(1 - x^2)y_0 + \varepsilon^2(1 - x^2)y_1 + O(\varepsilon^3) = 0, \quad \varepsilon \rightarrow 0.$$

Comparing terms in ε^0 gives

$$-4y_0''(x) = 0,$$

and so integrating twice, we find that

$$y_0(x) = Ax + B.$$

Applying boundary conditions $y_0(0) = 2$ and $y_0(1) = -\frac{1}{2}$ gives that $A = -\frac{5}{2}$ and $B = 2$.

Comparing terms in ε^1 gives

$$4y_1''(x) = (1 - x^2)y_0(x) = (1 - x^2)\left(2 - \frac{5}{2}x\right).$$

Integrating twice yields

$$y_1(x) = \frac{x^2}{4} - \frac{5x^3}{48} - \frac{x^4}{24} + \frac{x^5}{32} + Cx + D.$$

Application of homogeneous boundary conditions $y_1(0) = 0$ and $y_1(1) = 0$ gives that $D = 0$ and $C = \frac{-13}{96}$. Thus

$$y(x) = 2 - \frac{5}{2}x + \frac{1}{\mu^2} \left(\frac{x^5}{32} - \frac{x^4}{24} - \frac{5x^3}{48} + \frac{x^2}{4} - \frac{13x}{96} \right) + O(\mu^{-4}), \quad \mu \rightarrow \infty.$$