Solutions 1

2026

1. Prove that $5x^3 = O(x)$ as $x \to 0$.

   Solution: We note that for $0 < x < 1$, $|5x^3| \le |5x| = 5|x|$. It follows that $5x^3 = O(x)$, $x \to 0$.

2. Prove that $2x + 7 = o(x^2)$ as $x \to +\infty$.

   Solution: Let $c > 0$ be given. Then for $x > \frac{2+\sqrt{4+28c}}{2c}$, we have that

   $$|2x + 7| \le c|x^2|.$$

   (See Problem Class 2 solutions for a similar example with hints)

3. Let

   $$f(x) = x^2 \sin\frac{5}{x}, \qquad g(x) = 4x^2.$$

   (a) Prove that $f(x) = O(g(x))$ as $x \to 0$.
   (b) Show that the asymptotic formula $f(x) = o(g(x))$, as $x \to 0$, is invalid.
   (c) Is the asymptotic formula $g(x) = O(f(x))$ as $x \to 0$ valid?

   Solution:

   (a) We note that

   $$\left|x^2 \sin\frac{5}{x}\right| = \left|x^2\right| \left|\sin\frac{5}{x}\right| \le |x^2| \le 4|x^2| = |4x^2|$$

   for any $x \in \mathbb{R}$; it follows that $f(x) = O(g(x))$, $x \to 0$.

   (b) Consider the ratio

   $$r(x) = \left|\frac{f(x)}{g(x)}\right| = \frac{1}{4}\left|\sin\frac{5}{x}\right|.$$

   We note that $\lim_{x\to 0} r(x) = 0$ does not exist; it follows that $f(x) = o(g(x))$, as $x \to 0$, is an invalid asymptotic formula.

   (c) If the statement is true, then there exists some constant C such that in some neighbourhood of $x = 0$,

   $$|g(x)| \le C|f(x)|,$$

   or equivalently

   $$\left|\frac{g(x)}{f(x)}\right| = \left|\frac{4}{\sin(\frac{5}{x})}\right| \le C.$$

   It is not possible to find such a constant C, since there are infinitely many points near $x = 0$ (e.g. $x_n = \frac{5}{n\pi}$, $n = 1, 2, 3, \ldots$) at which $\sin\frac{5}{x} = 0$. At such points, $\frac{4}{\sin(\frac{5}{x})}$ becomes unbounded. It follows that the asymptotic formula is invalid.

4. Prove that $\ln x = O(x^{1/2})$ as $x \to +\infty$.

   Solution: For $x > 1$,

$$|\ln x| = \ln\left(\left(x^{1/2}\right)^2\right) = 2\ln x^{1/2} \leq 2x^{1/2} = 2|x^{1/2}|.$$

   It follows that $\ln x = O(x^{1/2})$, $x \to \infty$.

5. Obtain a two-term asymptotic approximation (i.e. an expression of the form $u(x) = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^2)$ as $\varepsilon \to 0$) of the solution to the second order regularly perturbed boundary value problem for the ordinary differential equation

$$\begin{aligned} &u''(x) - 3\varepsilon u(x) = 1 + \varepsilon, \quad 0 \leq x \leq 1 \\ &u(0) = 1, \qquad u(1) = 2, \end{aligned}$$

   where $\varepsilon > 0$ is a small parameter.

   Solution: We seek a solution in the form

$$u(x) = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^2), \quad \varepsilon \to 0.$$

   Substituting this Ansatz into the ODE, we obtain

$$u_0''(x) + \varepsilon u_1''(x) - 3\varepsilon u_0 + O(\varepsilon^2) = 1 + \varepsilon, \quad \varepsilon \to 0. \tag{1}$$

   Comparing terms in ε^0, we obtain the simple ODE $u_0'' = 1$, which we complement with the boundary conditions $u_0(0) = 1$ and $u_0(1) = 2$. By twice integrating and applying boundary conditions, the solution is given by

$$u_0(x) = \frac{x}{2}(x+1) + 1$$

   .

   Comparing terms in ε^1 in (1), we obtain the ODE $u_1''(x) - 3u_0(x) = 1$. That is,

$$u_1''(x) = 4 + \frac{3x^2}{2} + \frac{3x}{2},$$

   to which we associate the boundary conditions $u_1(0) = u_1(1) = 0$ (note these are homogeneous boundary conditions since the original boundary conditions have already been satisfied by the leading order term u_0). Solving this ODE by twice integrating and applying the boundary conditions, the solution is given by

$$u_1(x) = 2x^2 + \frac{x^4}{8} + \frac{3x^3}{12} - \frac{19x}{8}.$$

   The overall solution to the original problem is therefore

$$u(x) = \frac{x}{2}(x+1) + 1 + \varepsilon\left(2x^2 + \frac{x^4}{8} + \frac{3x^3}{12} - \frac{19x}{8}\right) + O(\varepsilon^2), \quad \varepsilon \to 0.$$

6. Obtain two-term asymptotic approximations of all three roots of the cubic equation

$$x^3 - x + \varepsilon = 0,$$

where $\varepsilon > 0$ is a small parameter.

Solution: We seek a solution in the form

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2), \quad \varepsilon \to 0.$$

Substituting this Ansatz into the cubic equation (noting that $x^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2)$), we obtain

$$x_0^3 + 3\varepsilon x_0^2 x_1 - x_0 - \varepsilon x_1 + \varepsilon + O(\varepsilon^2). \tag{2}$$

Comparing terms in ε^0, we obtain the problem $x_0^3 - x_0 = 0 \Leftrightarrow x_0(x_0^2 - 1) = 0$, and so $x_0 = 0, -1$ or $+1$.

Comparing terms in ε^1, we obtain the problem $3x_0^2 x_1 - x_1 + 1 = 0$. For the three cases $x_0 = 0, -1, +1$, this has respective solutions $x_1 = 1, -\frac{1}{2}, -\frac{1}{2}$. Our two-term approximations for the three roots are therefore

$$x = \begin{cases} 0 + \varepsilon + O(\varepsilon^2), \\ -1 - \frac{\varepsilon}{2} + O(\varepsilon^2), \\ +1 - \frac{\varepsilon}{2} + O(\varepsilon^2), \end{cases} \quad \varepsilon \to 0.$$

7. Let a_n be such that $a_{n+1} > a_n$. Prove that the sequence

$$\{\delta_n(x)\} = \{e^x x^{-a_n}\}, \quad x \to \infty,$$

for $n = 0, 1, 2, \ldots$, is an asymptotic sequence.

Solution: Consider the ratio $\frac{\delta_{n+1}(x)}{\delta_n(x)}$. We have that

$$\frac{\delta_{n+1}(x)}{\delta_n(x)} = \frac{e^x x^{-a_{n+1}}}{e^x x^{-a_n}} = x^{-(a_{n+1}-a_n)} \to 0 \text{ as } x \to \infty,$$

since $a_{n+1} - a_n > 0$. It follows that $\delta_{n+1}(x) = o(\delta_n(x))$ as $x \to \infty$, and so the sequence is an asymptotic sequence.