Problem Class 1 – Solutions

2026

Big-O

1. Prove that $4x^4 = O(x^3)$ as $x \to 0$.

   For $0 < x < 1$, we have that
   $$|4x^4| = 4x^4 \leq 4|x^3|.$$
   It follows that $4x^4 = O(x^3)$ as $x \to 0$.

2. Prove that $4x^4 = O(x^5)$ as $x \to \infty$.

   For $x > 1$, we have that
   $$|4x^4| = 4x^4 \leq 4|x|^5.$$
   It follows that $4x^4 = O(x^5)$ as $x \to \infty$.

3. Prove that $-4x^4 = O(x^5)$ as $x \to \infty$.

   For $x > 1$, we have that
   $$|-4x^4| = 4x^4 \leq 4|x|^5.$$
   It follows that $4x^4 = O(x^5)$ as $x \to \infty$.

4. Prove that $x^3 \cos \frac{3}{x} = O(7x^3)$ as $x \to \infty$.

   Note that for any $x > 0$, the following inequality holds:
   $$\left| x^3 \cos \frac{3}{x} \right| = |x^3| \left| \cos \frac{3}{x} \right| \leq |x^3| \cdot 1 = \frac{1}{7} |7x^3|.$$
   It follows that $x^3 \cos \frac{3}{x} = O(7x^3)$ as $x \to \infty$.

Little-O

5. Prove that $3x + 4 = o(x^2)$ as $x \to \infty$.

   Let $c > 0$ be given. Then for $x > \frac{3+\sqrt{9+16c}}{2c}$, we have that
   $$|3x + 4| \leq c|x^2|.$$
   It follows that $3x + 4 = o(x^2)$ as $x \to \infty$.

   Note: The key bit of this type of question is to find the critical value beyond which the inequality holds (in this case $\frac{3+\sqrt{9+16c}}{2c}$). Curve sketching the two functions concerned (here $3x + 4$ and cx^2) can be helpful.

6. Show that the following asymptotic formula is invalid: $x^3 \cos \frac{3}{x} = o(7x^3)$ as $x \to 0$.

   If $x^3 \cos \frac{3}{x} = o(7x^3)$ as $x \to 0$, then there exists some $\delta = \delta(c)$ for any given $c > 0$ such that $\left| \frac{x^3 \cos \frac{3}{x}}{7x^3} \right| \le c$ for all $0 < x < \delta$. However,

$$\left| \frac{x^3 \cos \frac{3}{x}}{7x^3} \right| = \frac{1}{7} \left| \cos \frac{3}{x} \right|,$$

   and since $|\cos \frac{3}{x}|$ oscillates rapidly between ± 1 as $x \to 0$, we note that $\lim_{x\to 0}\{\frac{1}{7}|\cos \frac{3}{x}|\}$ does not exist. We deduce that no such δ exists; it follows that $x^3 \cos \frac{3}{x} = o(7x^3)$ as $x \to 0$ is an invalid statement.

Regular perturbation problems

7. Obtain a three-term asymptotic approximation (i.e. an expression of the form $u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + O(\varepsilon^3)$ as $\varepsilon \to 0$) of the solution to the first order perturbed boundary value problem for the ordinary differential equation

$$u'(x) - \varepsilon u(x) = \sin x, \quad x > 0$$
$$u(0) = 1.$$

   where $\varepsilon > 0$ is a small parameter.

   We substitute the ansatz $u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + O(\varepsilon^3)$, $\varepsilon \to 0$ into the ODE to obtain

$$u_0'(x) + \varepsilon u_1'(x) + \varepsilon^2 u_2'(x) - \varepsilon u_0(x) - \varepsilon^2 u_1(x) + O(\varepsilon^3) = \sin x, \quad \varepsilon \to 0.$$

   Collecting coefficients of terms in ε^0 gives

$$u_0'(x) = \sin x,$$

   which upon integration with respect to x yields $u_0(x) = A - \cos x$, where A is an arbitrary constant. Applying the boundary condition $u_0(0) = 1$ gives that $A = 2$, so $u_0(x) = 2 - \cos x$.

   Collecting coefficients of terms in ε^1 gives

$$u_1'(x) - u_0(x) = 0,$$

   which upon substituting for the now known function u_0 becomes

$$u_1'(x) = 2 - \cos x.$$

   Integration with respect to x yields $u_1(x) = 2x - \sin x + B$, where B is an arbitrary constant. Applying the boundary condition $u_1(0) = 0$ (NB: this is zero NOT one, since terms after u_0 have homogeneous boundary conditions) gives that $B = 0$, so $u_1(x) = 2x - \sin x$.

   Collecting coefficients of terms in ε^2 gives

$$u_2'(x) - u_1(x) = 0,$$

   which upon substituting for the now known function u_1 becomes

$$u_2'(x) = 2x - \sin x.$$

Integration with respect to x yields $u_2(x) = x^2 + \cos x + C$, where C is an arbitrary constant. Applying the boundary condition $u_2(0) = 0$ gives that $C = -1$, so $u_2(x) = x^2 + \cos x - 1$.

Consequently the three-term asymptotic approximation is

$$u(x) = 2 - \cos x + \varepsilon(2x - \sin x) + \varepsilon^2(x^2 + \cos x - 1) + O(\varepsilon^3), \quad \varepsilon \to 0.$$

8. Obtain two-term asymptotic approximations of all three roots of the regularly perturbed cubic equation

$$x^3 - 3x + \varepsilon = 0,$$

where $\varepsilon > 0$ is a small parameter.

Problem moved to Assignment as we didn't cover it in the Problem Class. Hint: as usual, substitute in $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$ into the cubic equation and equate coefficients of ε^0 and then ε^1 to find x_0 and x_1 respectively.