Problem Class 2

2026

1. Consider the cubic equation
$$\varepsilon^2 x^3 + \varepsilon x^2 + 2x - 1 = 0,$$
where $\varepsilon > 0$ is a small parameter.

   (a) How many roots in $\mathbb{C}$ does this equation have?
   3 (by the fundamental theorem of algebra - it's a cubic).

   (b) Find three-term *regularly perturbed* solutions using the straightforward asymptotic expansion $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$, $\varepsilon \to 0$.
   Note first (thinking ahead to save us some effort!) that if we aim to eventually compare terms in $\varepsilon^0, \varepsilon^1, \varepsilon^2$, then we should write a one-term approximation for x^3 (because x^3 is multiplied by ε^2 in the given cubic equation) and a two-term approximation for x^2 (because x^2 is multiplied by ε in the given cubic equation). Therefore we note that
$$x^3 = x_0^3 + O(\varepsilon), \quad \varepsilon \to 0,$$
$$x^2 = x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2), \quad \varepsilon \to 0.$$
   Substitute these expressions into the cubic, compare coefficients of $\varepsilon^0, \varepsilon^1, \varepsilon^2$ and solve the resulting linear equations to deduce that $x_0 = \frac{1}{2}$, $x_1 = -\frac{1}{8}$ and $x_2 = 0$, whence the regularly perturbed root has asymptotic expansion
$$x = \frac{1}{2} - \frac{\varepsilon}{8} + O(\varepsilon^3), \quad \varepsilon \to 0.$$

   (c) Draw the Kruskal-Newton graph for this problem and use it to construct a singular perturbation asymptotic ansatz. See final page of solutions.

   (d) Use the singular perturbation ansatz from (d) to find two-term *singularly perturbed* solutions. Substitute the singular perturbation ansatz into the cubic equation to obtain
$$\varepsilon^{-1} z^3(\varepsilon) + \varepsilon^{-1} z^2(\varepsilon) + 2\varepsilon^{-1} z(\varepsilon) - 1 = 0.$$
   Multiply throughout by ε:
$$z^3(\varepsilon) + z^2(\varepsilon) + 2z(\varepsilon) - \varepsilon = 0. \tag{1}$$
   We now use regular perturbation methods on z. We make the regular perturbation ansatz
$$z(\varepsilon) = z_0(\varepsilon) + \varepsilon z_1(\varepsilon) + O(\varepsilon^2), \quad \varepsilon \to 0.$$
   Substituting this ansatz into (1) and comparing coefficients of terms in different degrees of ε gives
$$z_0^3 + z_0^2 + 2z_0 = 0, \tag{2}$$

and

$$3z_0^2 z_1 + 2z_0 z_1 + 2z_1 - 1 = 0. \tag{3}$$

Since $z_0 \neq 0$, we have that $z_0^2 + z_0 + 2 = 0$ and so

$$z_0^{\pm} = \frac{-1 \pm i\sqrt{7}}{2}.$$

Putting these values into (3) gives that the corresponding values for $z_1^{\pm}$ are

$$z_1^{+} = -\frac{1}{28}(7 - \sqrt{7}i)$$

and

$$z_1^{-} = -\frac{1}{28}(7 + \sqrt{7}i).$$

Thus the singularly perturbed roots $x^{\pm}$ are given by

$$x^{+} = \frac{1}{\varepsilon}\left(-\frac{1}{2} + i\frac{\sqrt{7}}{2} - \frac{\varepsilon}{28}(7 - \sqrt{7}i) + O(\varepsilon^2)\right), \quad \varepsilon \to 0,$$

and

$$x^{-} = \frac{1}{\varepsilon}\left(-\frac{1}{2} - i\frac{\sqrt{7}}{2} - \frac{\varepsilon}{28}(7 + \sqrt{7}i) + O(\varepsilon^2)\right), \quad \varepsilon \to 0.$$

2. For each of the following algebraic equations ((a)-(e)) which involve a small parameter $\varepsilon > 0$,

   (i) how many roots in $\mathbb{C}$ does the equation have?

   (ii) how many regularly perturbed roots in $\mathbb{C}$ does the equation have?

   (iii) how many singularly perturbed roots in $\mathbb{C}$ does the equation have?

   (a) $\varepsilon^2 x^3 + 3\varepsilon x^2 + x + 2 = 0$;

   (b) $x^5 + 4\varepsilon x^2 - 1 = 0$;

   (c) $\varepsilon^7 x^3 + \varepsilon x - 2 = 0$;

   (d) $\varepsilon^2 x^6 - \varepsilon^7 x^5 + 5\varepsilon^4 x^3 - 12\varepsilon x^2 + 5\varepsilon^4 x - 14 = 0$;.

   (e) $7\varepsilon^9 x^5 - 3\varepsilon^4 x^4 + \frac{3}{2}\varepsilon x^3 + 3x + 1 = 0$.

   For each of these, the number of roots in $\mathbb{C}$ is equal to the highest power of x (by the fundamental theorem of algebra). Considering the limit problem (putting $\varepsilon = 0$) – in particular how many roots it has – tells us how many regularly perturbed roots the original problem has. Any roots that are not regularly perturbed are singularly perturbed, so the answer to (iii) in each case is the answer to (i) minus the answer to (ii). Answers below are in the form (i) / (ii) / (iii):

   (a) 3 / 1 / 2;

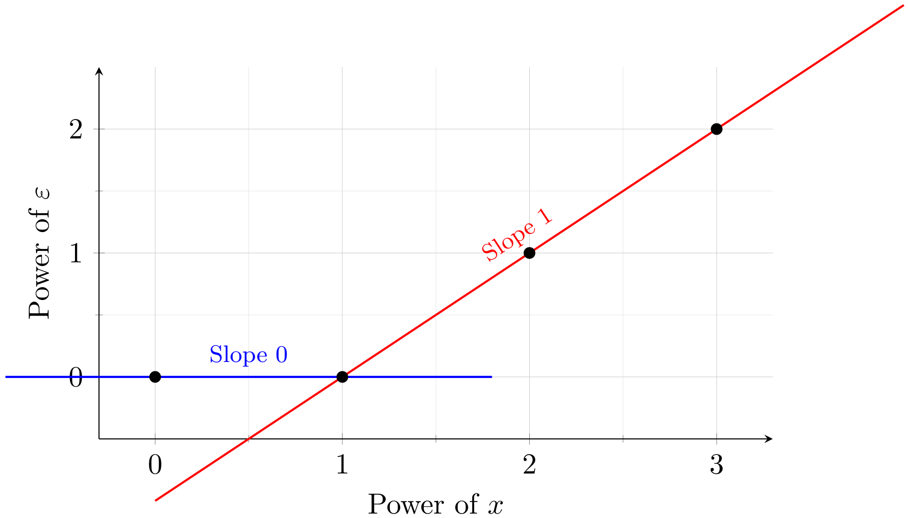
   (b) 5 / 5 / 0;

   (c) 3 / 0 / 3;

   (d) 6 / 0 / 6;

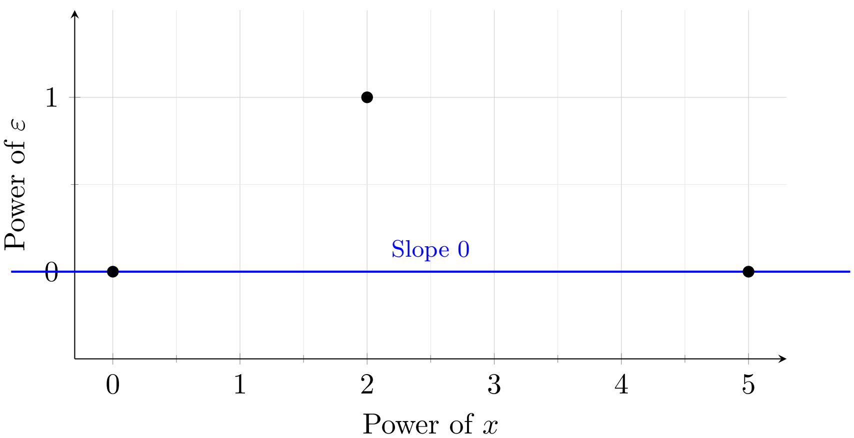
(e) 5 / 1 / 4.

3. (a)



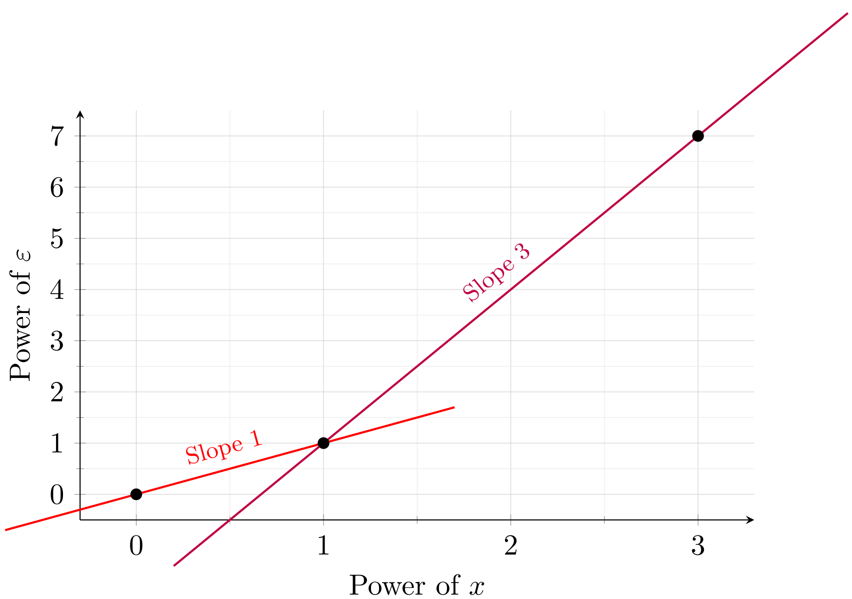
Power of ε
Power of x
Slope 1
Slope 0

(b)



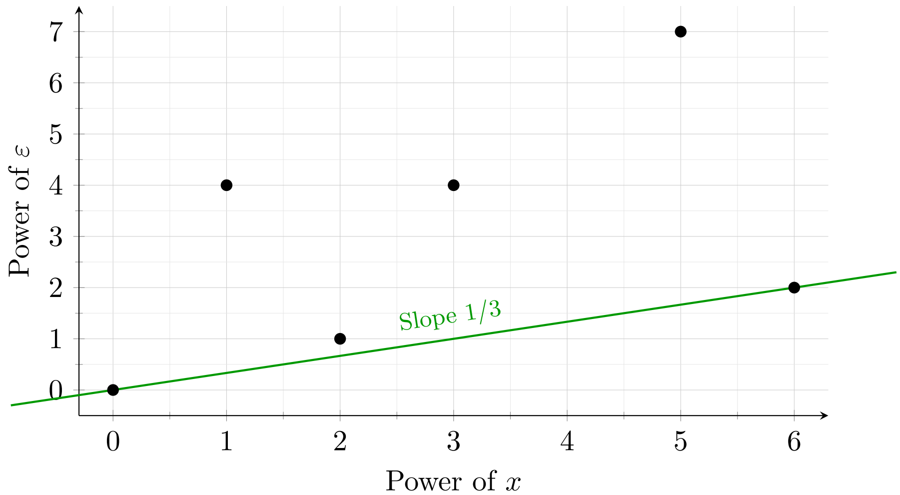
Power of ε
Power of x
Slope 0

(c)



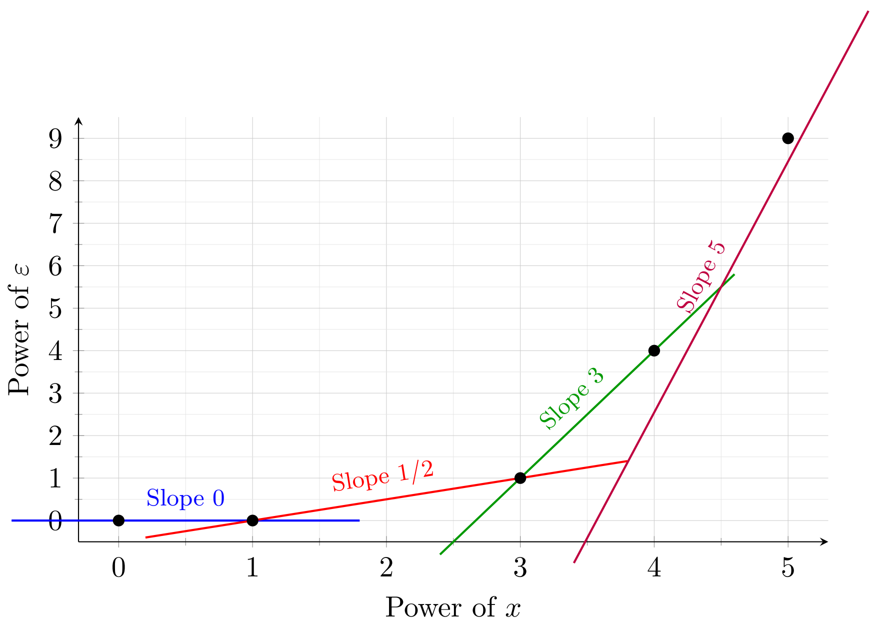
Power of ε
Power of x
Slope 3
Slope 1

(d)



Power of ε
Power of x
Slope 1/3

(e)



Power of ε
Power of x
Slope 0
Slope 1/2
Slope 3
Slope 5