Problem Class 2

2026

1. Consider the cubic equation
$$\varepsilon^2 x^3 + \varepsilon x^2 + 2x - 1 = 0,$$
where $\varepsilon > 0$ is a small parameter.

   (a) How many roots in $\mathbb{C}$ does this equation have?

   (b) Find three-term *regularly perturbed* solutions using the straightforward asymptotic expansion $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$, $\varepsilon \to 0$.

   (c) Draw the Kruskal-Newton graph for this problem and use it to construct a singular perturbation asymptotic ansatz.

   (d) Use the singular perturbation ansatz from (d) to find two-term *singularly perturbed* roots.

2. For each of the following algebraic equations ((a)-(e)) which involve a small parameter $\varepsilon > 0$,

   (i) how many roots in $\mathbb{C}$ does the equation have?

   (ii) how many regularly perturbed roots in $\mathbb{C}$ does the equation have?

   (iii) how many singularly perturbed roots in $\mathbb{C}$ does the equation have?

   (a) $\varepsilon^2 x^3 + 3\varepsilon x^2 + x + 2 = 0;$

   (b) $x^5 + 4\varepsilon x^2 - 1 = 0;$

   (c) $\varepsilon^7 x^3 + \varepsilon x - 2 = 0;$

   (d) $\varepsilon^2 x^6 - \varepsilon^7 x^5 + 5\varepsilon^4 x^3 - 12\varepsilon x^2 + 5\varepsilon^4 x - 14 = 0.$

   (e) $7\varepsilon^9 x^5 - 3\varepsilon^4 x^4 + \frac{3}{2}\varepsilon x^3 + 3x + 1 = 0$

3. Draw Kruskal-Newton graphs for all of the equations in Q2 (a)-(e) that you concluded were singularly perturbed. In what form(s) would you seek to approximate the roots for each of these problems?