Problem Class 1

A reminder of the definitions of big-O and little-o follows. Each definition covers three cases.

- **Definition [Big-O]:** Let f, g be real-valued functions. If there exist positive constants C and δ such that $|f(x)| \leq C|g(x)|$ for all $\begin{pmatrix} 0 < x < \delta \\ x > \delta \\ |x-a| < \delta \end{pmatrix}$, then we say $f(x) = O(g(x))$ as $\begin{pmatrix} x \to 0 \\ x \to \infty \\ x \to a \end{pmatrix}$.

- **Definition [Little-o]:** Let f, g be real-valued functions. If for every $c > 0$ there exists $\delta = \delta(c)$ such that $|f(x)| \leq c|g(x)|$ for all $\begin{pmatrix} 0 < x < \delta \\ x > \delta \\ |x-a| < \delta \end{pmatrix}$, then we say $f(x) = o(g(x))$ as $\begin{pmatrix} x \to 0 \\ x \to \infty \\ x \to a \end{pmatrix}$.

Note the difference between the two definitions: for big-O, the inequality $|f(x)| \leq C|g(x)|$ must hold for at least one value of C, whereas for little-o, the inequality $|f(x)| \leq C|g(x)|$ must hold for **every** $c > 0$.

Questions

Big-O

1. Prove that $4x^4 = O(x^3)$ as $x \to 0$.
2. Prove that $4x^4 = O(x^5)$ as $x \to \infty$.
3. Prove that $-4x^4 = O(x^5)$ as $x \to \infty$.
4. Prove that $x^3 \cos \frac{3}{x} = O(7x^3)$ as $x \to \infty$.

Little-O

5. Prove that $3x + 4 = o(x^2)$ as $x \to \infty$.
6. Show that the following asymptotoic formula is invalid: $x^3 \cos \frac{3}{x} = o(7x^3)$ as $x \to 0$.

Regular perturbation problems

7. Obtain a three-term asymptotic approximation (i.e. an expression of the form $u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + O(\varepsilon^3)$ as $\varepsilon \to 0$) of the solution to the first order perturbed boundary value problem for the ordinary differential equation

$$\begin{aligned} &u'(x) - \varepsilon u(x) = \sin x, \quad x > 0 \\ &u(0) = 1. \end{aligned}$$

   where $\varepsilon > 0$ is a small parameter.

   Hint: expand our unknown as a power series:

$$u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + O(\varepsilon^3), \quad \varepsilon \to 0,$$

   substitute into the ODE, compare coefficients of ε^0*, solve the resulting problem, then compare coefficients of* ε^1 *and so on.*

8. Obtain two-term asymptotic approximations of all three roots of the regularly perturbed cubic equation

$$x^3 - 3x + \varepsilon = 0,$$

   where $\varepsilon > 0$ is a small parameter.