

Assignment 2

2026

1. Consider the cubic equation

$$\varepsilon^2 x^3 + \varepsilon x^2 + 4x - 3 = 0.$$

- (a) How many roots in \mathbb{C} does this equation have?
- (b) Find two-term *regularly perturbed* solutions using the straightforward asymptotic expansion $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$, $\varepsilon \rightarrow 0$.
- (c) Draw the Kruskal-Newton graph for this problem and use it to construct a singular perturbation asymptotic Ansatz.
- (d) Use the singular perturbation Ansatz from (c) to find two-term *singularly perturbed* solutions.
- (e) Evaluate all three roots numerically for the case $\varepsilon = 0.01$.

2. Consider the singularly perturbed quartic equation

$$1 - x + 7\varepsilon x^2 - 3\varepsilon^3 x^3 + \varepsilon^7 x^4 = 0.$$

- (a) Find three-term *regularly perturbed* solutions using the straightforward asymptotic expansion.
- (b) Draw the Kruskal-Newton graph.
- (c) Which powers of ε would you use in the singular perturbation Ansatz for this problem? How many extra (i.e. singularly perturbed) roots would you expect to find?

3. Consider the ordinary differential equation

$$-4\mu^2 y''(x) + (1 - x^2)y(x) = 0,$$

with boundary conditions $y(0) = 2$ and $y(1) = -\frac{1}{2}$.

Find a two-term asymptotic approximation for the solution in the form

$$y(x) = y_0(x) + \mu^{-2}y_1(x) + O(\mu^{-4}), \quad \mu \rightarrow \infty,$$

assuming that $\mu \gg 1$ is a large positive parameter.*Hint: make the substitution $\varepsilon = \mu^{-2}$.*