Assignment 1

Starred questions due in via Blackboard by 6pm Tuesday 24th February 2026.

Landau Symbols

1. ⋆ Prove that $5x^3 = O(x)$ as $x \to 0$.

2. ⋆ Prove that $2x + 7 = o(x^2)$ as $x \to +\infty$

3. ⋆ Let
$$f(x) = x^2 \sin\frac{5}{x}, \qquad g(x) = 4x^2.$$
   (a) Prove that $f(x) = O(g(x))$ as $x \to 0$.
   (b) Show that the asymptotic formula $f(x) = o(g(x))$, as $x \to 0$, is invalid.
   (c) Is the asymptotic formula $g(x) = O(f(x))$ as $x \to 0$ valid?

4. Prove that $\ln x = O(x^{1/2})$ as $x \to +\infty$.
   You may use without proof the result that $\ln\alpha < \alpha$ for all $\alpha > 0$.

Regular Perturbation Problems

5. ⋆ Obtain a two-term asymptotic approximation (i.e. an expression of the form $u(x) = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^2)$ as $\varepsilon \to 0$) of the solution to the second order regularly perturbed boundary value problem for the ordinary differential equation
$$u''(x) - 3\varepsilon u(x) = 1 + \varepsilon, \quad 0 \le x \le 1$$
$$u(0) = 1, \qquad u(1) = 2,$$
   where $\varepsilon > 0$ is a small parameter.

6. ⋆ Obtain two-term asymptotic approximations of all three roots of the cubic equation
$$x^3 - x + \varepsilon = 0,$$
   where $\varepsilon > 0$ is a small parameter.

Asymptotic sequences

7. Let a_n be such that $a_{n+1} > a_n$. Prove that the sequence
$$\{\delta_n(x)\} = \{e^x x^{-a_n}\}, \quad x \to \infty,$$
   for $n = 0, 1, 2, \ldots$, is an asymptotic sequence.