PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2025

MA34210 – Asymptotic Methods in Mechanics

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

Formulae:
You may find the following Taylor-Maclaurin series and definite integral useful:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \text{converges for } x \in \mathbb{R}.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{converges for } x \in \mathbb{R}.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{converges for } |x| < 1.$$

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}.$$

Section A

1. Consider the following cubic equation, where $\varepsilon > 0$ is a small parameter:

$$x^3 - 6x^2 + 11x - 6 + 6\varepsilon = 0.$$

(a) Find all roots of the limit problem corresponding to $\varepsilon = 0$. [4 marks]

(b) Clearly explaining your reasoning, state whether the problem is a *regular* or *singular* perturbation of the limit problem corresponding to $\varepsilon = 0$. [3 marks]

(c) Obtain two-term asymptotic approximations (in the form $x_0 + \varepsilon x_1 + O(\varepsilon^2)$ as $\varepsilon \to 0$) of all three roots of the cubic equation. [10 marks]

2. (a) In algebraic problems, briefly explain why the regular perturbation ansatz

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3), \quad \varepsilon \to 0,$$

fails to approximate any singularly perturbed roots. [2 marks]

(b) For each of the following algebraic equations (i)-(iii) which all involve a small parameter $\varepsilon > 0$, state how many roots in $\mathbb{C}$ the equation has in total. Further, state how many of these roots are regularly perturbed and how many are singularly perturbed.

(i) $\varepsilon^2 x + 2 = 0$;

(ii) $x^5 + \varepsilon x - 1 = 0$;

(iii) $\varepsilon^2 x^3 - 2\varepsilon^7 x^2 + x - 3\varepsilon - 1 = 0$;

(iv) $\varepsilon x^3 + \varepsilon = \varepsilon x^3$. [2,2,2,2 marks]

3. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$g(x) = \int_0^x \frac{e^a \sin(2a)}{1+a}\, \mathrm{d}a.$$

(a) Obtain an asymptotic expansion for $g(x)$ as $x \to 0$, giving your answer in the form

$$g(x) = c_2 x^2 + c_3 x^3 + c_4 x^4 + O(x^5), \quad x \to 0,$$

where c_2, c_3, and c_4 are constants you should determine by using standard Taylor–Maclaurin series expansions. [7 marks]

(b) Hence or otherwise, state whether each of the following asymptotic statements is true or false (you are not required to justify your assertions):

(i) $g(x) \sim x^2, \quad x \to 0$;

(ii) $g(x) = o(x^2), \quad x \to 0$;

(iii) $g(x) = O(x^2), \quad x \to 0$. [3 marks]

4. (a) Give the formal definition of what is meant by the statement:

$$f(x) = O(g(x)), \quad x \to \infty.$$

[2 marks]

(b) Prove that $x^4 \cos\frac{5}{x} = O(x^4)$ as $x \to \infty$. [3 marks]

(c) Give the formal definition of what is meant by the statement:

$$f(x) = o(g(x)), \quad x \to \infty.$$

[2 marks]

(d) Prove that $3x + 4 = o(x^2)$ as $x \to \infty$. [4 marks]

(e) Give the formal definition of what is meant by the statement:

$$f(x) \sim g(x), \quad x \to 0.$$

[2 marks]

(f) State values of α and β so that the following asymptotic statement is valid:

$$\tan x \sim \alpha x^\beta, \quad x \to 0.$$

[2 marks]

5. Consider the following boundary value problem for a regularly perturbed ordinary differential equation (ODE), where $\varepsilon > 0$ is a small parameter:

$$u'(x) + 2\varepsilon\, u(x) = \sin x + \varepsilon\, \cos x, \quad x > 0;$$

with

$$u(0) = 0.$$

(a) Classify the ODE as either linear or non-linear, and state its order. [2 marks]

(b) Obtain a three-term asymptotic approximation of the solution in the form

$$u_a(x) = u_0(x) + \varepsilon\, u_1(x) + \varepsilon^2\, u_2(x) + O(\varepsilon^3), \quad \varepsilon \to 0.$$

[10 marks]

(c) Use a suitable (analytic, non-asymptotic) method to find an exact solution to the original problem and hence compute the numerical value of the relative error at $x = \pi/2$ in your approximation from part (b) for $\varepsilon = 0.05$, i.e. calculate

$$\left| \frac{u_a(\pi/2) - u(\pi/2)}{u(\pi/2)} \right|,$$

where u is the exact solution to the original problem.
Hint: you might find the following indefinite integrals useful:

$$\int \sin(x) e^{ax} \mathrm{d}x = \frac{e^{ax}}{a^2+1}(a \sin x - \cos x) + \text{constant},$$
$$\int \cos(x) e^{ax} \mathrm{d}x = \frac{e^{ax}}{a^2+1}(a \cos x + \sin x) + \text{constant}.$$

[6 marks]

Section B

6. This question concerns the Duffing equation, which describes the motion of an undamped point mass on a non-linear spring:

$$\ddot{x} + 3x + \varepsilon x^3 = 0.$$

(a) Explain what secular terms in asymptotic expansions are. Why are they usually undesirable in approximations describing physical phenomena? [3 marks]

(b) Clearly explaining your reasoning, use the Lindstedt-Poincaré method to find a two-term asymptotic approximation of a periodic solution of the Duffing equation above where the system is released from rest with an initial amplitude of A units and ε is a small dimensionless parameter. Also give a two-term expansion of the frequency of oscillations. [13 marks]

Hint: *You may find the following trigonometric identity useful:*

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta,$$

and that solutions to the differential equation $x''(t) + x(t) = A\cos(t) + B\cos(3t)$, $A, B \in \mathbb{R}$ *are of the form*

$$x(t) = \frac{A}{2}t\sin t - \frac{1}{8}B\cos(3t) + c_1\cos t + c_2\sin t.$$

(c) If the spring is *compliant*, then $\varepsilon < 0$. Physically, what is meant by saying that a spring is compliant? [2 marks]

(d) Describe briefly the effect upon the frequency of oscillations of a point mass on a compliant spring if the initial amplitude is increased. [2 marks]

7. Consider the following singularly perturbed cubic equation:

$$\varepsilon^2 x^3 + \varepsilon x^2 - 2x + 1 = 0.$$

(a) Draw the Kruskal-Newton graph for this problem and use it to construct a singular perturbation asymptotic ansatz. [6 marks]

(b) Use the singular perturbation ansatz to find two-term approximations of both singularly perturbed roots (you are not required to approximate the regularly perturbed root in this question). [7 marks]

SECTION B CONTINUES ON NEXT PAGE

8. For odd $k \in \mathbb{N}$, $k = 2n-1$ for some $n \in \mathbb{N}$. The double factorial of an odd natural number k (denoted $k!!$) is defined by

$$k!! = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{2^n}{\sqrt{\pi}} \, \Gamma\left(n + \frac{1}{2}\right).$$

Here, Γ denotes the gamma function, defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t.$$

Furthermore, note that

$$\Gamma\left(n + \frac{1}{2}\right) = \left(n + \frac{1}{2}\right)^{n+\frac{1}{2}} \int_0^\infty e^{-\left(n+\frac{1}{2}\right)(z - \ln z)} \, \mathrm{d}z.$$

Clearly explaining each step in your working, use Laplace's method to find a leading order approximation for $(2n-1)!!$ as $n \to \infty$ in the form

$$(2n-1)!! \sim \alpha(\beta n)^{\gamma n},$$

where α, β and γ are constants you should determine.

Hint: you may find it useful to recall that

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

[10 marks]

9. Determine the range of values of a for which

$$\sinh(x) - \sin(x) = O(x^a), \quad x \to 0^+,$$

justifying your answer fully. [7 marks]

END OF PAPER