1872 PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI-MEHEFIN / MAY-JUNE 2023

MA34210 - Asymptotic Methods in Mechanics

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX83 neu FX85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Ar ôl eistedd, gall myfyrwyr lenwi tudalen flaen y llyfryn atebion a'r papur presenoldeb.

Peidiwch ag agor y papur arholiad tan y dywedir wrthych am wneud hynny.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX83 or FX85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

Once seated, students may complete the front cover of the answer book and the attendance slip.

Do not open the question paper until instructed to do so.

You may find the following Taylor-Maclaurin series and definite integrals useful:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}.$$

Section A

1. (a) Give the formal definition of what is meant by the statement: $f(x) = O(g(x))$ as $x \to \infty$. [2 marks]

(b) Prove that $x^2 \cos \frac{2}{x} = O(x^2)$ as $x \to \infty$. [3 marks]

(c) Give the formal definition of what is meant by the statement: $f(x) = o(g(x))$ as $x \to \infty$. [2 marks]

(d) Prove that $x + 2 = o(x^2)$ as $x \to \infty$. [4 marks]

(e) Give the formal definition of what is meant by the statement: $f(x) \sim g(x)$ as $x \to \infty$. [2 marks]

(f) State a function $a(x) \neq e^x$ for which $a(x) \sim e^x$ as $x \to \infty$. [2 marks]

(g) State the largest value of n for which the following asymptotic statement is valid:

$$x^5 \sin x = O(x^n), \quad x \to 0^+.$$

[2 marks]

2. Consider the following cubic equation, where ε is a small parameter:

$$x^3 - 3x^2 - 4x + \varepsilon = 0.$$

(a) Find all roots of the limit problem corresponding to $\varepsilon = 0$. [4 marks]

(b) Clearly explaining your reasoning, state whether the problem is a *regular* or *singular* perturbation of the limit problem corresponding to $\varepsilon = 0$. [3 marks]

(c) Obtain two-term asymptotic approximations (in the form $x_0 + \varepsilon x_1 + O(\varepsilon^2)$ as $\varepsilon \to 0$) of all three roots of the cubic equation. [10 marks]

3. (a) Explain what is meant by saying that a problem involving a small parameter is singularly perturbed. [2 marks]

(b) In algebraic problems, briefly explain why the regular perturbation ansatz $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$ fails to approximate any singularly perturbed roots. [2 marks]

(c) For each of the following algebraic equations ((i)-(iii)) which all involve a small parameter $\varepsilon > 0$, state how many roots in $\mathbb{C}$ the equation has in total. Further, state how many of these roots are regularly perturbed and how many are singularly perturbed.

(i) $\varepsilon^3 x + \varepsilon = 1$;

(ii) $x^5 + \varepsilon x^2 - 5 = 0$;

(iii) $\varepsilon^3 x^3 - 2\varepsilon^2 x^2 + x - 2\varepsilon - 1 = 0$. [2,2,2 marks]

4. Consider the following function which is defined by an integral:

$$f(x) = \int_0^x \frac{e^z \cos z}{2 - z} \mathrm{d}z.$$

(a) Obtain an asymptotic expansion for $f(x)$ as $x \to 0$, giving your answer in the form

$$f(x) = a_1 x + a_2 x^2 + a_3 x^3 + O(x^4), \quad x \to 0,$$

where a_1, a_2 and a_3 are constants you should determine. [10 marks]

(b) Hence or otherwise, state whether each of the following asymptotic statements is true or false (you are not required to justify your assertions):

(i) $f(x) \sim \dfrac{x}{2} \cos x, \quad x \to 0;$

(ii) $f(x) = O(x), \quad x \to 0;$

(iii) $f(x) = o(x), \quad x \to 0.$ [3 marks]

5. Consider the following boundary value problem for a regularly perturbed ordinary differential equation (ODE), in which $\varepsilon > 0$ is a small parameter:

$$\begin{cases} u'(x) - 2\varepsilon u(x) = 1 + \varepsilon \cos x, \quad x > 0; \\ u(0) = 1. \end{cases} \tag{1}$$

(a) Classify the ODE as either linear or non-linear, and state its order. [2 marks]

(b) Obtain a three-term asymptotic approximation of the solution to boundary value problem (1), i.e. obtain an approximation of the form

$$u_a(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 u_2(x) + O(\varepsilon^3), \ \varepsilon \to 0.$$

[11 marks]

Section B

6. Recall that for $n \in \mathbb{N}$, $n! = \Gamma(n+1)$ and that $\Gamma(n+1)$ can be written in the following form:

$$\Gamma(n+1) = n^{n+1} \int_0^\infty e^{-n(z-\ln z)} \mathrm{d}z.$$

Clearly explaining each step in your working, use Laplace's method to find a leading order approximation for $\Gamma(n+1)$ and hence derive Stirling's approximation:

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}, \quad n \to \infty.$$

[12 marks]

7. Consider the following singularly perturbed cubic equation:

$$\varepsilon^2 x^3 - 5\varepsilon x^2 + 6x + 1 = 0.$$

(a) Draw the Kruskal-Newton graph for this problem and use it to list ansatzes that could be used to approximate all roots for small ε. [4 marks]

(b) Use the ansatzes found in (a) to find two-term approximations of all roots of the cubic equation. [7 marks]

8. A self-excited oscillator can be modelled by the differential equation

$$x''(t) + \varepsilon(x^2(t) - 1)x'(t) + 4x(t) = 0, \tag{2}$$

where $\varepsilon > 0$ is a small parameter.

(a) Use regular perturbation methods to find a two-term approximation to a solution of (2) with initial conditions $x(0) = a_0$ and $x'(0) = 0$ for some $a_0 > 0$. The hint given at the end of this question may be of use. [9 marks]

(b) State what secular terms are, identify any such terms in your two-term approximation from (a), and briefly outline why secular terms are undesirable when modelling an oscillator. [3 marks]

(c) Use the Lindstedt-Poincaré method to find a two-term asymptotic approximation of a non-trivial periodic solution of (2) with the same initial conditions as described in (a). Hence deduce the frequency and value of a_0 required to obtain periodic oscillations. Again, the hint given at the end of this question may be of use. [15 marks]

Hint: for $A, B, C \in \mathbb{R}$, the general solutions of the ODEs

$$x''(t) + 4x(t) = A\cos(2t) + B\sin(2t) + C\cos^2(2t)\sin(2t), \text{ and}$$

$$y''(t) + y(t) = A\cos(t) + B\sin(t) + C\cos^2(t)\sin(t)$$

are given by

$$x(t) = c_1\cos(2t) + c_2\sin(2t) - \frac{4B+C}{16}t\cos(2t) + \frac{A}{4}t\sin(2t) - \frac{C}{128}\sin(6t), \text{ and}$$

$$y(t) = c_3\cos(t) + c_4\sin(t) - \frac{4B+C}{8}t\cos(t) + \frac{A}{2}t\sin(t) - \frac{C}{32}\sin(3t),$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.