PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2022

MA34210 - Asymptotic Methods in Mechanics

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Adam Vellender, on asv2@aber.ac.uk.

You should write out solutions to the paper and upload them to Blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Formulae:
You may find the following Taylor-Maclaurin series and definite integrals useful:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \text{for all } x.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}.$$

Section A

1. (a) Prove that $x^5 \cos x = O(x^4)$ as $x \to 0$. [3 marks]

(b) Prove that $1 + 3x = o(x^2)$ as $x \to \infty$. [4 marks]

(c) Let $\psi(x) = 6x^{-5}e^{-2/x}$. Write down a function $\phi(x)$, not equal to ψ, for which $\phi(x) \sim \psi(x)$ as $x \to \infty$. [1 mark]

2. Obtain two-term asymptotic approximations (in the form $x_0 + \varepsilon x_1 + O(\varepsilon^2)$ as $\varepsilon \to 0$) of all three roots of the regularly perturbed cubic equation

$$x^3 - 3x^2 - 13x + 15 + \varepsilon = 0,$$

where $\varepsilon > 0$ is a small parameter. [12 marks]

3. Consider the following boundary value problem for a regularly perturbed ordinary differential equation, in which $\varepsilon > 0$ is a small parameter:

$$\begin{cases} u''(x) - 12\varepsilon u(x) = (x + \varepsilon)^2, \quad x > 0; \\ u(0) = 0, \qquad u(1) = 1. \end{cases} \tag{1}$$

(a) State the order of the ODE, and whether it is linear or non-linear. [2 marks]

(b) Obtain a two-term asymptotic approximation of the solution to the boundary value problem (1). [12 marks]

4. Consider the quintic equation

$$8 + 7x - x^2 + 2\varepsilon x^3 - 5\varepsilon^3 x^4 + 2\varepsilon^8 x^5 = 0,$$

where $\varepsilon > 0$ is a small parameter.

(a) How many roots in $\mathbb{C}$ does the quintic equation have? [1 mark]

(b) Write down the limit problem (the equation corresponding to $\varepsilon = 0$). How many roots in $\mathbb{C}$ does it have? [2 marks]

(c) Hence state (briefly explaining your reasoning) whether the quintic equation is regularly or singularly perturbed. [3 marks]

(d) Draw the Kruskal-Newton graph for this problem. [5 marks]

(e) In what forms would you seek to approximate the roots for this problem? How many roots would you expect each ansatz to yield? [5 marks]

(f) Which ansatz would allow you approximate the root that diverges most rapidly as $\varepsilon \to 0$? [1 mark]

(g) Obtain leading order (i.e. one-term) asymptotic approximations for all roots as $\varepsilon \to 0$. [8 marks]

5. Obtain an asymptotic expansion for the integral

$$\mathcal{I}(x) = \int_0^x \frac{t(\sin(2t) + e^t)}{1 - t} \mathrm{d}t,$$

as $x \to 0$, giving your answer in the form

$$\mathcal{I}(x) = c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + O(x^5), \quad x \to 0,$$

where c_1, c_2, c_3 and c_4 are constants to be determined. [11 marks]

Section B

6. This question concerns the Duffing equation, which describes the motion of an undamped point mass on a non-linear spring:

$$\ddot{x} + 3x + \varepsilon x^3 = 0. \tag{2}$$

(a) Explain what secular terms in asymptotic expansions are. Why are they usually undesirable in approximations describing physical phenomena? [3 marks]

(b) Clearly explaining your reasoning, use the Lindstedt-Poincaré method to find a two-term asymptotic approximation of a periodic solution of the Duffing equation (2) where the system is released from rest with an initial amplitude of A units and ε is a small dimensionless parameter. Also give a two-term expansion of the frequency of oscillations. [13 marks]

Hint: *You may find the following trigonometric identity useful:*

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta,$$

and that solutions to the differential equation $x''(t) + x(t) = A\cos(t) + B\cos(3t)$, $A, B \in \mathbb{R}$ *are of the form*

$$x(t) = \frac{A}{2}t\sin t - \frac{1}{8}B\cos(3t) + c_1\cos t + c_2\sin t.$$

(c) If the spring is *compliant*, then $\varepsilon < 0$. Physically, what is meant by saying that a spring is compliant? [2 marks]

(d) Describe briefly the effect upon the frequency of oscillations of a point mass on a compliant spring if the initial amplitude is increased. [2 marks]

7. Consider the integral

$$\mathcal{P}(\lambda) = \int_{-1}^{1} f(t)e^{-\lambda g(t)}\mathrm{d}t$$

where the functions f and g are given by

$$f(t) = \cos t, \qquad g(t) = 2 + \sin^2 t.$$

(a) For which value of t does the function $g(t)$ take its minimum value over $(-1, 1)$? What is the value of $g(t)$ at that point? [2 marks]

(b) Using Laplace's method, perform a sequence of approximations to demonstrate that

$$\mathcal{P}(\lambda) \sim c\lambda^{-1/2}e^{-2\lambda}, \quad \lambda \to +\infty,$$

where c is a constant you should determine. Give brief explanations of the approximations used throughout your working. [10 marks]

(c) Briefly explaining your reasoning, would Laplace's method allow you to approximate the integral

$$\mathcal{R}(\lambda) = \int_{-1}^{4} f(t)e^{-\lambda g(t)}\mathrm{d}t,$$

where f and g remain the same functions as earlier in the question? [3 marks]

8. The following ordinary differential equation models an over-damped harmonic oscillator:

$$\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0. \tag{3}$$

(a) Find the exact general solution to the limit problem corresponding to $\varepsilon = 0$. [4 marks]

(b) Explain why the limit problem is not a good approximation to the initial value problem in which an over-damped harmonic oscillator is released from its equilibrium position at an initial velocity v_0. [2 marks]

(c) Using a suitable rescaling of the time variable, construct a two-term inner asymptotic expansion for the problem (3) with the initial conditions described in (b). [9 marks]